Recursive Construction of the Minimal Prime Digraphs

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ALGOS 2020

August 28, 2020
I have known Maurice Pouzet since the first semester of the academic year 1991-1992, during which, as a DEA student at Claude Bernard University (Lyon 1), I took his course in ”Theory of Relations”. He has helped and supported me a lot since that beginning. His permanent support was crucial for me throughout my career and in particular for the creation and functioning of the Combinatorics Team of the University of Sfax (Tunisia) as well as of the Research Group in Discrete Mathematics of King Saud University (Saudi Arabia). I must say that in addition to his well-known scientific qualities, Maurice has a very distinguished human quality.

On behalf of all my colleagues from these two aforementioned teams, I say: ”Thank you very much dear Maurice !”.
In a digraph $D$, a module is a vertex subset $M$ such that every vertex outside $M$ does not distinguish the vertices in $M$. A digraph $D$ with more than two vertices is prime if the empty set, the singleton sets, and the full set of vertices are the only modules in $D$. A prime digraph $D$ is $k$-minimal if there is some $k$-element vertex subset $U$ such that no proper induced subdigraph of $D$ containing $U$ is prime.

In this paper, we give a recursive procedure to construct the minimal prime digraphs.
All digraphs mentioned here are finite, and have no loops and no multiple edges.
Thus a digraph (or directed graph) $D$ consists of a nonempty and finite set $V(D)$ of vertices with a collection $E(D)$ of ordered pairs of distinct vertices, called the set of edges of $D$. Such a digraph is denoted by $(V(D), E(D))$.
Recall that the subdigraph of a digraph $D$ induced by a nonempty vertex subset $X$ is denoted by $D[X]$, and if $|V(D)| \geq 2$, then for each vertex $x$, the subdigraph $D[V(D) \setminus \{x\}]$ is also denoted by $D - x$.
Given a digraph $D$, the underlying graph of $D$ is denoted by $\tilde{D}$, and the complement of $D$, defined by $V(\overline{D}) = V(D)$ and $E(\overline{D}) = (V(D)^2 \setminus \{(x, x) : x \in V(D)\}) \setminus E(D)$, is denoted by $\overline{D}$. 
Let $D$ be a digraph.
A vertex subset $M$ is a *module* (or a clan or an interval or an autonomous set) of $D$ if every vertex outside $M$ does not distinguish the vertices in $M$.
The empty set, the singleton sets, and the full set of vertices are *trivial modules*.
A digraph is *indecomposable* if all its modules are trivial; otherwise it is *decomposable*.
Indecomposable digraphs with at least three vertices are *prime digraphs*. 
A partition $\mathcal{P}$ of the vertex set $V(D)$ of $D$ is a *modular partition* of $D$ if all its elements are modules of $D$. It follows that the elements of $\mathcal{P}$ may be considered as the vertices of a new digraph, the *quotient* of $D$ by $\mathcal{P}$, denoted by $D/\mathcal{P}$, and defined on $\mathcal{P}$ as follows: for any distinct elements $X$ and $Y$ of $\mathcal{P}$, $XY \in E(D/\mathcal{P})$ if $xy \in E(D)$ for any $x$ and $y$ with $x \in X$ and $y \in Y$.

A vertex subset $X$ is a *strong module* of $D$ provided that $X$ is a module of $D$, and for every module $Y$ of $D$, if $X \cap Y \neq \emptyset$, then either $X \subseteq Y$ or $Y \subseteq X$.

If $|V(D)| \geq 2$, then $\mathcal{P}(D)$ denotes the set of maximal, strong modules of $D$, under the inclusion, among the strong modules of $D$ distinct from $V(D)$. 

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The following theorem gives Gallai’s decomposition result.

**Theorem 1 (T. Gallai 1967).**

Let $D$ be a digraph with more than one vertex. The set $\mathcal{P}(D)$ is a modular partition of $D$, and the quotient $D/\mathcal{P}(D)$ is prime, or an acyclic tournament, or a complete digraph, or an empty digraph.

**Frame of a digraph.**

Given a digraph $D$ with more than one vertex, the quotient $D/\mathcal{P}(D)$ is the *frame* of $D$. 
Minimal digraph.

A prime digraph $D$ is *minimal* for a nonempty vertex subset $A$ if each proper induced subdigraph of $D$ containing $A$ is decomposable. In this case, we say that the digraph $D$ is $A$-minimal.

Given a positive integer $k$, a digraph $D$ is $k$-minimal if it is minimal for some $k$-set of vertices.

Remark (Minimal digraphs are unavoidable induced subdigraphs).

Given a prime digraph $D$, for each vertex subset $B$, there is a vertex subset $C$ including $B$ such that the subdigraph $D[C]$ is $B$-minimal.

This concept was introduced by A. Cournier and P. Ille (1998). They characterized the 1-minimal and 2-minimal digraphs. M. Alzohairi and Y. Boudabbous (2014) described the 3-minimal triangle-free graphs. Finally, M. Alzohairi (2015) described the triangle-free graphs which are minimal for some nonstable 4-vertex subset.
In this paper, given an integer $k$, with $k \geq 3$, we give a method for constructing the $k$-minimal prime digraphs from the $(k - 1)$-minimal prime digraphs.

To do so, we introduce the following notion which is a refinement of the notion of a chain introduced by M. Chudnovsky, R. Kim, S.-I. Oum, and P. Seymour (2016).

**Strong chain.**

Let $D$ be a digraph and $p$ be an integer with $p \geq 2$. A sequence $(s_0, s_1, ..., s_p)$ of distinct vertices is a strong chain of $D$ if $\{s_j : j < i\}$ is a module of $D - s_i$, for each element $i$ of $[2, p]$. 
First we establish the following **separation principle** which plays a crucial role in the proof of our main result.

**Proposition 1 (Separation principle).**

Let $D$ be a prime digraph which is $A$-minimal, where $A$ is a vertex subset with $|A| \geq 2$, and $X$ be a vertex subset including $A$ such that the frame of the subdigraph $D[X]$ is prime and $\mathcal{P}(D[X])$ has a unique non-singleton element $M$. Assume that there are two distinct vertices $s_0$ and $s_1$ in $A$ such that $M \cap A = \{s_0, s_1\}$ and if $|M| \geq 3$, then the vertices in $M$ form a strong chain $(s_0, s_1, ..., s_p)$ of $D$.

Then there is a vertex $u$ outside $X$ such that $(s_0, s_1, ..., s_p, u)$ is a strong chain of $D$. Moreover, either $V(D) = X \cup \{u\}$ or the frame of $D[X \cup \{u\}]$ is prime with $M \cup \{u\} \in \mathcal{P}(D[X \cup \{u\}])$.

The key of the proof is the study of the prime subdigraphs (resp. the subdigraphs with prime frames) in a prime digraph, obtained by A. Ehrenfeucht and G. Rozenberg (1990) (resp. Y. Boudabbous and P. Ille (2010)).
Second we obtain our main result.

**Theorem 2.**

Let $D$ be a prime digraph, and $A$ be a $k$-element vertex subset with $k \geq 3$. If the digraph $D$ is $A$-minimal, then at least one of the following assertions holds.

1. There is a vertex $x$ in $A$ such that $D$ or $D - x$ is $(A \setminus \{x\})$-minimal.
2. There are distinct vertices $x$ and $y$ in $A$ such that $D$ is obtained from some $(A \setminus \{x\})$-minimal digraph $H$, with $x \not\in V(H)$, by adding $x$ and a sequence $(s_1, \ldots, s_m)$ of distinct vertices outside $V(H) \cup \{x\}$ with $m \geq 1$ such that $(y, x, s_1, \ldots, s_m)$ is a strong chain of $D$.

Moreover, if $m \geq 2$, then either $V(H)$ is a module of $D[V(H) \cup \{s_m\}]$, or there is a vertex $u$ in $A \setminus \{x, y\}$ such that $\{u, s_m\}$ is a module of $D[V(H) \cup \{s_m\}]$, or $D[V(H) \cup \{s_m\}]$ is $((A \setminus \{x\}) \cup \{s_m\})$-minimal.
(3) The subset $A$ is a stable set of an element $W$ of $\{D, \overline{D}\}$, the elements of $A$ are pendant vertices of the underlying graph $\widetilde{W}$ of $W$, the corresponding edges of which form a matching in $\widetilde{W}$, there are a vertex $x$ in $A$ and an $(A \setminus \{x\})$-minimal digraph $H$ with $x \notin V(H)$, and there is a vertex $u$ outside $V(H) \cup \{x\}$ such that $\widetilde{W}[V \setminus V(H)]$ is a path $P$ with ends $x$ and $u$ and $\widetilde{W}$ is obtained from the union of $\widetilde{H}$ and $P$ by adding a nonempty set of edges between $u$ and $V(H) \setminus A$.

**Remark 1.**

Notice that the digraphs constructed in the three assertions are all prime, but they are not necessarily $A$-minimal. Their minimality depends on the structure of the digraph $H$. 

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Remark 2.

In Assertion 2, \( \{x, y\} \cup \{s_j, j < m\} \) is a module of \( G - s_m \) and the different possibilities of the behavior of the vertex \( s_m \) with respect to the vertices of \( H \) are well specified.
We illustrate the three assertions by the following graphs which are $A$-minimal with $A = \{x, y, z, w\}$.

(a) Assertion (1)  
(b) Assertion (3)  
(c) Assertion (2) with $m=1$ and $s$ is adjacent to none of $V(H)$

(d) Assertion (2) with $m=1$ and $\{s, w\}$ is a module of $G[V(H) \cup \{s\}]$

(e) Assertion (2) with $m=1$ and $G[V(H) \cup \{s\}]$ is prime
Consider a prime digraph $D := (V, E)$ and a $k$-element vertex subset $A$, with $k \geq 3$, such that $D$ is $A$-minimal.

**Assume** that the first assertion of Theorem 2 is not satisfied. Thus, given a vertex $x$ in $A$ and a vertex subset $W$, if the subdigraph $D[W]$ is $(A - \{x\})$-minimal, then $x \notin W$ and the subdigraph $D[W \cup \{x\}]$ is decomposable, and hence by the partition’s theorem of A. Ehrenfeucht and G. Rozenberg (1990), either $W$ is a module of $D[W \cup \{x\}]$ or there is a vertex $y$ in $W$ such that $\{x, y\}$ is a module of $D[W \cup \{x\}]$.

**Step 1.** Assume that there are a vertex $x$ in $A$, a vertex subset $W$ such that $x \notin W$ and the subdigraph $D[W]$ is $(A \setminus \{x\})$-minimal, and a vertex $y$ of $W$ such that $\{x, y\}$ is a module of $D[W \cup \{x\}]$.

By a technical lemma based on the separation principle, we prove that the second assertion of Theorem 2 is satisfied.
Step 2. Thus by the partition’s theorem of A. Ehrenfeucht and G. Rozenberg (1990), we may assume that $x \notin W$ and $W$ is a module of $D[W \cup \{x\}]$, for each vertex $x$ in $A$, and for each vertex subset $W$ such that the subdigraph $D[W]$ is $(A \setminus \{x\})$-minimal.

In this case, $A \setminus \{u\}$ is a module of the subdigraph $D[A]$, for each vertex $u$ of $D[A]$, and hence we can deduce that $D[A]$ is an empty digraph or a complete digraph. By interchanging $D$ and $\overline{D}$, if necessary, we will assume that $D[A]$ is an empty digraph.
First, we obtain the following claim.

**Claim 1.**
For each vertex $x$ in $A$, $d_{\tilde{D}}(x) = 1$.

Second, consider a vertex $x$ in $A$ and a vertex subset $X$ such that the subgraph $H := D[X]$ is $(A - \{x\})$-minimal. Recall that $x \notin X$ and $X$ is a module of $D[X \cup \{x\}]$. Since the digraph $D$ is prime, the graph $\tilde{D}$ is connected. Let $p := \min\{d_{\tilde{D}}(x, t) : t \in X\}$, $u_0$ be a vertex in $X$ with $d_{\tilde{D}}(x, u_0) = p$, and $u_0, u_1, ..., u_p$ be an $u_0x$-path in $\tilde{D}$. Clearly, $p \geq 2$, $\{u_1, ..., u_p\} \cap X = \phi$, and $N_{\tilde{D}}(u_i) \cap X = \phi$, for each element $i$ of $[2, p]$.

We prove that the third assertion of Theorem 2 is satisfied.

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Abstract

Introduction

Sketch of the proof of Theorem 2.

References


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