Well quasi-ordering and embedabbility of relational structures;
with an emphasis on problems and conjectures

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ALGOS-2000, Loria, Nancy

August 27, 2020
A relational structure $R$ is embeddable into a relational structure $R'$ and we set $R \leq R'$ if $R$ is isomorphic to some induced substructure of $R'$. Embeddability is a quasi-order on the class of relational structures. In the late forties, Fraïssé, following the work of Cantor, Hausdorff and Sierpinski, highlighted the role of this quasi-order in the theory of relations. For example, he conjectured that the class of countable chains is well quasi ordered (w.q.o.) under embeddability, a conjecture positively solved by Laver in the early seventies with the use of the theory of better quasi ordering (b.q.o.), a far reaching strengthening of the notion of w.q.o., invented by Nash-Williams.
Fraïssé also noted that basic notions about ordered sets (posets) like initial segments, ideals, chains and antichains have a direct counterpart in terms of relational structures. During the last fifty years the approach of classes of relational structures by means of posets has slowly developed. In this talk, I will briefly some salient facts. I will put emphasis on problems and conjectures on well-quasi-ordering and hereditary classes of relational structures. Some of these problems go back to the seventies.
Most of the questions discussed here can be formulated in terms of graphs (undirected, with no loops). For several reasons, I consider structures which are more general.

A relational structure is a realization of a language whose non-logical symbols are predicates. This is a pair $R := (V, (\rho_i)_{i \in I})$ made of a set $V$ and of a family of $m_i$-ary relations $\rho_i$ on $V$.

The set $V$ is the domain or base of $R$, we set $V(R)$ for the base of $R$. The family $\mu := (m_i)_{i \in I}$ is the signature of $R$.

The substructure induced by $R$ on a subset $A$ of $V$, simply called the restriction of $R$ to $A$, is the relational structure $R_{|A} := (A, (A^{m_i} \cap \rho_i)_{i \in I})$.

Notions of isomorphism and local isomorphism from a relational structure to an other one are defined in a natural way as well as the notion of isomorphic type.
A relational structure $R$ is embeddable into a relational structure $R'$ and we set $R \leq R'$ if $R$ is isomorphic to some restriction of $R'$. Embeddability is a quasi-order on the class of relational structures. If $R$ is embeddable in $R'$ and $R'$ is embeddable in $R$, then $R$ and $R'$ are equimorphic or siblings. If $R$ and $R'$ are isomorphic there are trivially equimorphic. The converse holds if one of the two is finite.

The problem of deciding if two finite graphs are isomorphic is one of the most challenging problem of complexity theory. Thomassé (circa 2000) has asked how many siblings (up to isomorphism) an infinite relation may have, conjecturing that if it have a finite number then it has just one. In his talk, Claude Laflamme reported some progress on this conjecture.
As noted by Fraïssé, basic notions about ordered sets, like initial segments, ideals, chains and antichains have a direct counterpart in terms of relational structures. For example, a class $\mathcal{C}$ of structures is hereditary if it contains every structure which can be embedded into some member of $\mathcal{C}$. Clearly, hereditary classes are initial segments of the class of relational structures quasi-ordered by embeddability. If $R$ is a relational structure, the age of $R$ is the set $\text{Age}(R)$ of finite restrictions of $R$ considered up to isomorphy (a set introduced by R. Fraïssé). This is an ideal of the poset made of finite structures considered up to isomorphy and ordered via embeddability. As shown by Fraïssé, 1948, every countable ideal has this form.
Examples

Hereditary classes of finite graphs and posets abound in combinatorics (e.g. classes of comparability graphs; planar graphs, $n$-colourable graphs; permutation graphs, perfect graphs; classes of serie-parallel posets, posets coverable by $n$ chains; posets of Dushnik-Miller dimension at most $n$, permutation graphs (alias comparability graphs of dimension 2 posets ). For the logician, hereditary classes of finite structures (of a given finite arity) are classes of finite models of universal theories (theories axiomatized by universal sentences). Such classes have a finite axiomatization if and only they have only finitely many bounds (counted up to isomorphism) Tarski-Vaught’s Theorem, 1957.
A **bound** of a hereditary class $C$ of finite relational structures is a finite relational structure $S$ which does not belong to $C$ whereas every smaller $S'$ belongs. The set $Bound(C)$ of bounds determines $C$. The bounds form an antichain. If $B$ is an antichain, then $Forb(B)$, the set of finite $S$ embedding no member of $B$, is a hereditary class and $B$ is the boundary of $Forb(B)$.

If $B$ is an antichain, every subset $B'$ determines a hereditary class, namely $For(B')$. Thus an infinite antichain determines $2^{\aleph_0}$ hereditary class. W.r.t. embeddability there are plenty of infinite antichains. E.g. among graphs, finite cycles. Or paths with decorated ends, e.g., double-ended forks (see Figure 1).
The previous observations illustrate the importance of antichains in the study of hereditary classes; e.g., classes for which the boundary is finite, classes containing no infinite antichains. Well-quasi-order, one of the most important notion in the theory of ordered sets, come right here. But, as a matter of fact, its importance in the study of finite structures did not come with embeddability but rather with the minor quasi-ordering, and the theorem of Robertson and Seymour (circa 1980). Its importance for infinite structures came with Laver’s theorem on scattered chains (1970).
Introduced to Fraïssé’s conjectures and to the theory of well quasi order and better quasi ordering at the end of the sixties, by Ernest Corominas, my master, I thought that there was an interesting line of research between posets and relational structures and on the role of wqo and bqo. I will briefly survey some of the results and problems that I found interesting.

I will present first the notion of well quasi order (wqo), then I will discuss, the properties of wqo hereditary classes. Then will come the preservation of wqo under addition of unary predicates. I will continue with uniformly prehomogeneous structures. I will conclude with extensions of Laver’s theorem.
Basic notions.
Well foundation, Well-quasi-order, Height

A poset $P$ is **well founded** if every non-empty subset has some minimal element. It is **well-quasi-ordered**, w.q.o. for short, if every non-empty subset contains finitely many minimal elements (this number being non-zero). This last notion was introduced independently by Higman (who refer to Kaplanski) and Erdős and Rado in the fifties.

A final segment $F$ of a poset $P$ is **finitely generated** if for some finite subset $K$ of $P$, $F$ equals the set $\uparrow K := \{ y \in P : x \leq y \text{ for some } x \in K \}$.

Let $P$ be a well founded poset; the **height** $h(x, P)$ of an element $x \in P$ is an ordinal defined by induction by the formula:

$$h(x, P) = \text{Sup}\{ h(y, P) + 1 : y \in P, y < x \}.$$
An important result on w.q.o. is de Jongh-Parikh theorem (1977).

**Theorem**

*If a poset $P$ is w.q.o. then all the linear extensions of $P$ are well-ordered and there is one having the largest possible order type.*

This largest order type, denoted $o(P)$, is the *ordinal length* of $P$. For example, if $Q$ is a w.q.o. then $o(Q) = h(Q, I(Q))$ where $I(Q)$ is the set of initial segments of $Q$ (Zaguia, 1983, Pouzet-Zaguia, 1985) (this is an equivalent formulation of de Jongh-Parikh’s theorem).
Basic notions.

Ideals

An ideal of a poset $P$ is any non empty initial segment of $P$ which is up-directed. The poset $P$ is wqo iff it is a finite union of ideals which are wqo (this is a consequence of a result of Erdös and Tarski (1942)).

There is a relationship between the ordinal length of $P$ and the height of the maximal ideals of $P$. For an example, if $P$ is wqo and up-directed then $o(P) = h(P, \mathbf{I}(P)) \leq \omega^{H(P)}$ where $H(P) := h(P, \mathbf{J}(P))$ and $\mathbf{J}(P)$ is the set of ideals of $P$ (Nejib Zaguia, 1983).

The ordinal length of several posets have been computed. For example, the ordinal length of the direct sum, resp. product, of two posets is the Heissenberg sum, resp. product, of their ordinal length (Carruth (1946), de Jongh and Parikh (1977)); if $A^*$ is the set of words over un alphabet $A$ made of $k$ letters then $o(A^*) = \omega^{\omega^{k-1}}$ (de Jongh and Parikh (1977)), this formula was extended to an arbitrary wqo (see Schmidt (1978)); the ordinal length of the collection of binary trees is the ordinal $\epsilon_0$ (Schmidt (1978)); for more, see her habilitation and Rathjen and Weiermann (1993)).
Some observations:
A hereditary class $C$ made of finite structure, is w.q.o. iff its antichains are finite.
A hereditary class $C$ made of finite structure, is w.q.o. iff it is a finite union of wqo ages about poset.
If a hereditary class $C$ is wqo, it is countable and thus $o(C)$ is a countable ordinal.

Problem

*Is every countable ordinal attained by some hereditary class of finite structures with a finite signature? Is there a largest countable ordinal depending upon the signature?*

*What about hereditary classes of graphs?*

If we allow unbounded signature, the answer to the problem above is negative: every ordinal below $\omega_1$ can be attained.
Wqo ages

If it is obvious that there are uncountably many hereditary classes, it is not obvious the number of wqo classes is the same. This reduces to see that there are uncountably many wqo ages. Note that an ideal of finite structures is wqo iff its antichains are finite; hence if it is wqo it is countable and thus this is the age of some relational structure. In 1978, I proved that existed $2^{\aleph_0}$ wqo ages of birelations $(\mathbb{N}, c, u)$ where $c$ is the consecutivity relation on $\mathbb{N}$ and $u$ a unary relation. Such $u$ play a role in symbolic dynamic under the name of uniformly recurrent sequences. In his theses (1992, 2002), Sobrani got the same conclusion with graphs (undirected with no loops). With Imed Zaguia, we go further: A graph $G$ is path-minimal if it contains induced paths of finite unbounded length and every induced subgraph $G'$ of $G$ with the same property embeds a copy of $G$.

Theorem

There are $2^{\aleph_0}$ path-minimal graphs whose ages are pairwise incomparable and wqo.
With Sobrani, we proved:

**Theorem**

Let $\mathcal{A}$ be an age. If $\mathcal{J}(\mathcal{A})$ is w.q.o then $o(\mathcal{A}) = \omega^\alpha \cdot q$ where $\alpha$ is such that $\omega \cdot \alpha \leq H(\mathcal{A}) < \omega \cdot (\alpha + 1)$ and $q$ is the number of ages included into $\mathcal{A}$ whose height is between $\omega \cdot \alpha$ and $\omega \cdot (\alpha + 1)$.

Examples say more. Ages of height $\omega^2$ are w.q.o. and in fact better-quasi-ordred (see below). They may have infinitely many bounds. In fact, there are $2^{\aleph_0}$ ages of undirected graphs of height $\omega^2$. There are larger ordinals. The ordinal length of the class of finite forest is $\epsilon_0$. The ordinal length of the class of finite series-parallel poset is $\Gamma_0$ (Sobrani, I, 2003).
In the previous theorem, we needed the condition that $J(A)$ is w.q.o. This is related to the notion of better quasi-ordering and a question of Nash-Williams.

A poset $P$ is better quasi-ordered (b.q.o.) if the class $P^{<\omega_1}$ of countable ordinal sequences is w.q.o under embeddability of sequences (alternatively, the transfinite iterates $I^\alpha(P)$, for ordinals $\alpha$, of the set of initial segments of $P$ are all w.q.o. Nash-Williams, who invented this notion and proved the fundamental results, conjectured that natural classes of structures which are w.q.o are in fact b.q.o.

**Problem**

*Is a hereditary class of finite structures with finite signature b.q.o. whenever it is w.q.o.?*
The answer is negative if the signature is infinite and the arity is unbounded. There are w.q.o. ages which are not b.q.o. This is a bit technical and I refer to a paper with Sobrani 2001. The result we got is this:

**Theorem**

Let $P$ be a poset. If $P$ embeds into $[\omega]^{<\omega}$ then there are two ages $A$ and $B$ with $A$ totally ordered and $A \subseteq B$ such that the set $D(A, B)$ of ages between $A$ and $B$ is isomorphic to $I(P)$.
A central question from early seventies

Let $\mathcal{A}$ be the age of a countable structure; Since $\mathcal{A}$ is countable, ages included into $\mathcal{A}$ coincide with ideals. Let $D(\mathcal{A})$ be the collection of ages included into $\mathcal{A}$. The following implications $(i) \Rightarrow (ii) \iff (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$ hold:

- $\mathcal{A}$ is w.q.o.;
- $D(\mathcal{A})$ is countable;
- $D(\mathcal{A})$ is topologically scattered;
- Chains in $D(\mathcal{A})$ are at most countable;
- $D(\mathcal{A})$ is well-founded;
- The poset $[\omega]<\omega$ of finite subsets of $\omega$ does not embed into $\mathcal{A}$.

Problem

Is the reverse implication hold when the signature is finite? What about ages of graphs?, of ordered graphs?, of permutations?
Is there a combinatorial characterization of hereditary classes?

Consider the simpler case of an age. Let $R$ on a set $V$. Say that two finite subsets $A, A'$ of $V$ are **equivalent** if there is an isomorphism of the structure induced by $R$ on $A$ onto the structure induced by $R'$ on $A'$. This gives a partition of the finite subsets of $V$ then an ordering on the parts: A part $\tau$ is less than $\tau'$ if there are $A \subseteq A'$ such that $\tau$ is the class of $A$ and $\tau'$ the class of $A'$ (this amounts to every $A'$ contains some $A$).

We can see that as an ordered quotient of the powerset $\mathcal{P}(V)$.

**Problem.** Characterize combinatorially these quotients.

Clearly, the members of each class have the same cardinality, next they have the same **frequency vector**: given $A'$, the frequency vector $\chi_{A'}$ will be made of a sequence of numbers, counting for each class, how many members of that class are included into $A'$. In general, this does not suffice. But all properties of classes of relational structures that I can prove use only that.

This appeared in a paper with Ivo Rosenberg on Sperner property for groups and relations. For more, see the lecture by Dwight Duffus.
A problem of Abraham, Bonnet and Kubis on bqo

Abraham, Bonnet and Kubis asked the following question

**Question**

Is a wqo a countable union of bqo’s?

We may ask for more

**Question**

If a poset \( P \) has no infinite antichain, is its domain a countable union of sets \( A_n \) such that the order on each \( A_n \) is a strengthening of a finite intersection of linear orders?

If \( P \) is well founded then the \( A_n \) are wqo, hence \( P \) is a countable union of bqo’s. This would answer positively ABK’s question.

Try to use induction on the rank \( r(P) \) of \( P \), where \( r(P) \) is the height of the empty set in \( A(P) \) the poset of antichains of \( P \) ordered by reverse of inclusion. Avraham has proved that if \( r(P) < \omega_1^2 \) then \( P \) is a countable union of chains.
These question point to the class $\mathcal{P}$ of posets $P$ which are countable unions of strengthening of finite dimensional posets. Why strengthening? A strengthening of $\omega_1 \times \omega_1$ is not necessarily a countable union of finite dimensional posets (use Erdos-Tarski). These question lead Abraham, Bonnet and Dzamonja to look at the indivisibility of wqo. A poset $P$ is $\kappa$-indivisible if $P$ embeds in some block of any partition in less that $\kappa$ blocks. If ABK question has a positive answer, a $\omega_1$-indivisible wqo must be a bqo. At this moment, very few $\omega_1$-indivisible wqos are known.
The profile of a relational structure $R$ is the function $\phi_R$ which counts for every integer $n$ the number $\phi_R(n)$ of substructures of $R$ induced on the $n$-element subsets, isomorphic substructures being identified. Clearly, this function only depends upon the age $\text{Age}(R)$ of $R$. Lectures by Falque, Oudrar and Thiéry concentrate on this notion. I will just recall few things.

First: The profile of an infinite relational structure is non-decreasing. The result was conjectured with Fraïssé. The proof, based on Ramsey theorem, appears in Fraïssé’s Cours de Logique 1971 (Exercice 8, p.113). The result was improved in 1976 by showing that $\phi_R(n) \leq \phi_R(n + p)$ provided that the domain of $R$, possibly finite, has at least $2n + p$ elements. The proof is based on the non-degeneracy of the Kneser matrix of the $n$-element subsets of a $2n + p$ element set, a result obtained independently by Gottlieb 1966 and Kantor 1972.
Next, Profiles of relational structures with $\mu$ finite have either a polynomial growth or their growth is faster than every polynomial. The growth of $\varphi_R$ is polynomial of degree $k$ if $a \leq \frac{\varphi_R(n)}{n^k} \leq b$ where $a$ and $b$ are non-negative constants; the growth is faster than every polynomial if $\frac{\varphi_R(n)}{n^k}$ goes to infinity with $n$ for every $k$).

The result is more precise:

**Theorem**

Let $\mathcal{C}$ be a hereditary class of finite structures with finite signature. The profile of $\mathcal{C}$ is bounded by a polynomial iff $\mathcal{C}$ is w.q.o. and $o(\mathcal{C}) < \omega^{\omega}$ (in fact, $o(\mathcal{C}) = \omega^n.k + \nu$ avec $\nu < \omega^n$ iff the growth of the profile is polynomial with degree $n - 1$).
Use the fact that the age $\mathcal{A}$ of a relational structure $R$ with finite kernel has height $\omega \cdot n + p$ iff its profile grows like a polynomial of degree $n - 1$. Furthermore, $\mathcal{A}$ has finitely many bounds and the collection of subages of $\mathcal{A}$ is w.q.o. (in fact $\mathcal{A}$ is the age of an almost multichainable structure). Hence, we may apply a formula obtained with Sobrani.

**Theorem**

*If the age of a relational structure $R$ is inexhaustible and $\varphi_R$ has polynomial growth of degree $k$ then*

$$\varphi_R(n) \leq \binom{n + k}{k}$$

*for every integer $n$.***
Problem

The profile has a polynomial growth of degree $k$ if and only if the generating series of the profile $H_{\varphi_R} := \sum \varphi_R(n)$ is a rational fraction of the form:

$$\frac{P(Z)}{(1 - Z)(1 - Z^2)\cdots(1 - Z^{k+1})}.$$

where $P$ is a polynomial with integer coefficients (possibly negative). This amounts to say that $\varphi_R$ is, for $n$ large enough, a quasi-polynomial, that is a polynomial in $n$ with periodic coefficients; with the fact that $\varphi_R$ is non-decreasing this will give that $\varphi_R(n) \sim an^k$, a fact not know yet.

With Thiéry (2013) we got a positive answer for relational structures admitting a finite decomposition into monomorphic components; several important structures, eg tournaments with polynomial profiles (e.g., see Boudabbous and I, 2010) have this property, but not graphs. The case of graphs was settled independently by Balogh et al in 2006. They prove also that if the profile is not polynomial then it is above the partition function.
Balogh, Bollobás and Morris proved in 2006 that if $\mathcal{C}$ is a hereditary class of finite ordered graphs then its profile $\varphi_\mathcal{C}$ (by extension, the function which counts for every integer $n$ the numbers of $n$-element structures in that class) is either polynomial or is ranked by the Fibonacci functions (see Klazar 2010 for recent developments). Using the notion of monomorphic decomposition, Oudrar proves that the class $\mathcal{S}$ of ordered binary structures which do not have a finite monomorphic decomposition has a finite basis (a subset $\mathcal{B}$ such that every member of $\mathcal{S}$ embeds some member of $\mathcal{B}$). In the case of ordered reflexive directed graphs, the basis has 1242 members and the profile of their ages grows at least as the Fibonacci function. From this, she get a dichotomy result. Her result was extended in a recent work to ordered relational structures (see her lecture).

**Problem**

*Is the profile of a relational structure $R$ bounded by some exponential whenever the age of $R$ is well-quasi-ordered under embeddability?*
Preserving w.q.o. by adding a linear order

The concatenation of two relational structure $R$ and $S$ on the same domain is the relational structure, denoted by $R \circ S$, made of the relations of $R$ followed by the relations of $S$.

**Problem**

1. Let $C$ be a hereditary class of finite structures. If $C$ quasi-ordered under embeddability is w.q.o., is it true that for every $R \in C$ there is some linear order $L_R$ on $V(R)$, the domain of $R$, such that the class of $R \circ L_R$ is w.q.o.?

2. Let $R$ be a relational structure. If $\text{Age}(R)$ is w.q.o. is $\text{Age}(R \circ L)$ w.q.o. for some linear order on $V(R)$?
(2) ⇒ (1). If $\mathcal{C}$ is w.q.o. then $\mathcal{C}$ is a finite union of ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$. Each $\mathcal{I}_i$ is countable (for each integer $m$, members of size $m$ form an antichain, since $\mathcal{I}_i$ is wqo such an antichain must be finite) thus is the age of some relational structure $R_i$. If (2) has a positive answer, there is some linear order $L_i$ such that $\text{Age}(R_i.L_i)$ is wqo. Then, $\bigcup_{i=1,n} \text{Age}(R_i.L_i)$ is wqo.

No clear that (1) ⇒ (2):
Let $R$ and let $\mathcal{C}$ be the class of $R'.L'$ such that $R' \in \text{Age}(R)$ and $L'$ is a linear order on $V(R')$. We may find a wqo subclass $\mathcal{C}'$ such that $\text{Age}(R) := \{R' : R'.L' \in \mathcal{C}'\}$. If $\mathcal{C}'$ is hereditary, compactness theorem will allow to find $L$ on $R$ such that $\text{Age}(R.L)$ is wqo. But, if $\mathcal{C}'$ is not hereditary?.
Let $P$ be a poset. A **labelling** of a relational structure $R$ by $P$ is a map $f$ from $V(R)$, the domain of $R$, into $P$. Let $C$ be a class of relational structures of a fixed signature $\mu$. Denote by $P^C$ the class of relational structures $R \in C$ labelled by $P$, that is the class of pairs $(R, f)$ with $f : V(R) \to P$. We quasi order $P^C$ by **dominance** : $(R, f) \leq (R', f')$ if there is an embedding $h : R \to R'$ such that $f(x) \leq f'(h(x))$ for every $x \in V(R)$. 

Examples

Suppose that $P = \{0, 1\}$ with 0 incomparable to 1. Then $P^C$ with this quasi-order is isomorphic to the collection $C_1$ made of the collection of relational structures quasi-ordered by embeddability which is made of concatenats $R.U$ where $R \in C$ and $U$ is a unary relation on $V(R)$. More generally, let $m$ be an integer and $P$ be an antichain of size $2^m$ then $P^C$ with the quasi-order above is isomorphic to the collection $C_m$ made of the collection of relational structures quasi-ordered by embeddability and made of concatenats $R.U_1 \ldots U_m$ where $R \in C$ and $U_1, \ldots, U_m$ are $m$ unary relation on $V(R)$. Indeed, let us identify the $2^m$-element antichain $P$ with $\mathcal{P}(\{1, \ldots, m\})$, the power set of $\{1, \ldots, m\}$, ordered by the equality relation. To $f : V \to \mathcal{P}(\{1, \ldots, m\})$ associate $U(f) := (U_1, \ldots U_m)$ where $U_i := \{x \in V(R) : i \in f(x)\}$. Clearly $(R, f) \leq (R', f')$ iff $R.U(f) \leq R'.U(f')$. More interestingly, order $\mathcal{P}(\{1, \ldots, m\})$ with the inclusion order, in this case $R.U_1 \ldots U_m \leq R'.U'_1 \ldots U'_m$ mean that there is some embedding $h : R \to R'$ such that $x \in U_i$ implies $h(x) \in U'_i$ for every $x \in V(R)$ and $i := 1, \ldots, m$. 
Let $R$ be a relational structure and $m$ be a non-negative integer. A labelling of $R$ with $m$ constants is the structure $R$ with $m$ distinguished elements $a_1, \ldots, a_m$ of $V(R)$ added. An embedding of $(R, a_1, \ldots a_m)$ into $(R', a'_1, \ldots a'_m)$ is an embedding $f$ of $R$ into $R'$ such that $f(a_i) = a'_i$ for $i = 1, \ldots m$. If $C$ is a class of structures, we denote by $C.m^{-}$ the class of $(R, a_1, \ldots a_m)$ where $R \in C$.

**Problem**

Let $C$ be a hereditary class of finite structures. If $C.2^{-}$ is wqo, is $P^{C}$ wqo for every w.q.o $P$?

Let $P$ be a w.q.o. We say that a class $C$ of structures is $P$-wqo if $P^{C}$ is wqo. A class $C$ of relational structures is very-well-quasi-ordered, in short v.w.q.o., if for every integer $m$ the class $C_{m}$ made of $R \in C$ added of $m$ unary relations is w.q.o for the embeddability relation. Equivalently, $C$ is $P$-wqo for every finite poset $P$. We say that $C$ is hereditarily w.q.o., resp. hereditarily b.q.o. if $P^{C}$ is w.q.o., resp b.q.o., for every w.q.o., resp. b.q.o. $P$. 
Theorem

Let $C$ be a class of finite relational structures, then:

1. $C$ is v.w.q.o iff $\downarrow C$, its downward closure, is v.w.q.o.
2. If $\downarrow C$ is v.w.q.o. then all the ages it contains are almost inexhaustible.
3. If $\downarrow C$ is v.w.q.o. and all of its members have the same finite signature $\mu$ then it has only finitely many bounds, (I. 1972).
The questions above were dormant for almost fifty years. A progress was made recently for permutation graphs (comparability graphs of intersection of two linear orderings). With Imed Zaguia we proved this:

**Theorem**

*For a hereditary class $C$ of finite bipartite permutation graphs the following properties are equivalent:

1. There is some nonnegative integer $k$ such that the path $P_k$ on $k$ vertices does not belong to $C$;
2. $C$ has only finitely many bounds;
3. $C$ is hereditary w.q.o.;
4. The class consisting of the elements of $C$ labelled by two constants is w.q.o.*
Homogeneous structures and their automorphism groups are subject to an intensive research. For an example, see the lectures of Nesetril, Hubicka, Sauer. Following the work of Kechris, Pestov and Todorcevic (2005), homogeneous structures on which an linear order can be added to produce an homogeneous structure have received a lot of attention.

A notion related to homogeneity is this: a structure $R$ is uniformly prehomogeneous (u.p.h.) if for every finite set $F$ of the domain $V$ of $R$ there is a finite superset $F'$ of $F$ whose cardinality in bounded by some function $\theta$ of the cardinality of $F$ such that every local isomorphism of $R$ defined on $F$ extends to an automorphism provided that it extends to $F'$.

**Problem**

Let $R$ be a uniformly prehomogeneous structure. If $\text{Age}(R)$ w.q.o. is $(R, L)$ u.p.h. and $\text{age}(R, L)$ w.q.o. for some linear order $L$ on $V(R)$?
A group $G$ acting on a set $V$ is **oligomorphic** if for each integer $n$ the number of orbits of the action of $G$ on $n$-element subsets of $V$ is finite. As shown by Ryll-Nardzewski, the automorphism group of a countable structure $R$ is oligomorphic iff $R$ is $\aleph_0$-categorical. Saracino and I obtained independently:

**Theorem**

A countable relational structure $R$ such that $\text{Aut}(R)$ is oligomorphic is equimorphic to a countable $R'$ which is uniformly prehomogeneous and furthermore $\text{Aut}(R')$ is oligomorphic.

**Theorem**

Let $\mathcal{C}.m^-$ be the collection of $(S, a_1, \ldots, a_m)$ where $S \in \mathcal{C}$. If $\mathcal{C}.m^-$ is wqo then there is a uniformly prehomogeneous structure $R$ with age $\mathcal{C}$. 
Hint. Fix an integer $m$. Since $C \cdot m^-$ is w.q.o., it is a finite union of ideals. This is the test I gave in 1972 for the existence of a uniformly prehomogeneous structure $R$ with age $C$.

If the profile of an ideal $C$ of finite structures is bounded by a polynomial, $P^C$ is wqo for every wqo, hence there is some uniformly prehomogeneous structure whose age is $C$.

I would be tempting to conjecture that the orbital profile of the automorphism group is polynomial, but, as noted by Thiéry, this is not true.
Extension of Laver’s theorem

Laver proved that the class of chains which are countable unions of scattered chains is b.q.o. hence w.q.o. under embeddability. Fraïssé asked how to extend this result to ages. There are bqo ages of graphs whose the collection of countable structures with this given age is not wqo (I found some examples in 1978, other were given in Sobrani thesis).

A relational structure is binary if it consists of relations which are unary or binary. If \( R := (V, (\rho_i)_{i \in I}) \) is binary, a subset \( A \) of \( V \) is autonomous if \( \rho_i(x, y) = \rho_i(x', y) \) and \( \rho_i(y, x) = \rho_i(y, x') \) for all \( i \in I, x, x' \in A \) and \( y \in V \setminus A \). A binary structure is indecomposable if all its autonomous subsets are trivial (that is are either \( \emptyset \), singletons, or the whole set).

Delhommé [16] and independently Mckay [31] showed:

**Theorem**

*If a hereditary class \( \mathcal{C} \) of finite binary structures of finite arity contains only finitely many indecomposable members then the class of countable structures whose age is included into \( \mathcal{C} \) is b.q.o.*
This result extends Thomassé’s result (1999)[61] on the w.q.o. character of the class of countable series-parallel posets, which extends the famous Laver’s theorem (1971)[27] on the w.q.o. character of the class of countable chains.

With Oudrar, we conjecture:

**Problem**

*If the set $\text{Ind}(\mathcal{C})$ of indecomposable structures included into a hereditary class $\mathcal{C}$ of finite structures is hereditary b.q.o then the class of countable structures whose age is included into $\mathcal{C}$ is b.q.o.*
Suppose that a poset $P$ is well founded with no infinite antichain. Is it possible to prove that the set of initial segments of $P$ is well founded without the axiom of dependent choices?
Thank you for your attention.


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