Ordering infinities

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August 27, 2020  In honour of the 75th birthday of Maurice Pouzet
Back to the dark unordered ages

Georg Cantor
Georg Cantor

Cardinal numbers
Back to the dark unordered ages

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Cardinal numbers

Ordinal numbers

0, 1, 2, ...
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Cardinal numbers

Ordinal numbers

$0, 1, 2, \ldots, \omega$
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Cardinal numbers

Ordinal numbers

$0, 1, 2, \ldots, \omega, \omega + 1, \ldots$
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Cardinal numbers

Ordinal numbers

\[ 0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots \]
Georg Cantor

Cardinal numbers

Ordinal numbers

\[0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2\]
Georg Cantor

Cardinal numbers

$0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2, \ldots, \omega^3$
Georg Cantor

Cardinal numbers

\[0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega\]
Georg Cantor

Cardinal numbers

Ordinal numbers

\[ 0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \mathbb{N}_1, \ldots \]
Georg Cantor

Cardinal numbers

$0, 1, 2, ..., \omega, \omega + 1, ..., \omega \cdot 2, \omega \cdot 2 + 1, ..., \omega^2, ..., \omega^3, ..., \omega^\omega, ..., N_1, ...$

Ordinal numbers

$\omega^\omega + 2 \cdot 3 + \omega^8 \cdot 7 + \omega \cdot 3 + 2 \cdot 9 + \omega^\omega + 1 \cdot 3 + \omega^7 \cdot 5 + \omega^8 + \omega^2 \cdot 111 + 2020$
Paul du Bois-Reymond
Paul du Bois-Reymond

Precursor of asymptotic calculus

\[ \log x < \frac{x}{2} < \frac{x^2}{10} \quad (x \to \infty) \]
Paul du Bois-Reymond

Precursor of asymptotic calculus

\[
\log x < \frac{x}{2} < \frac{x^2}{10} \quad (x \to \infty)
\]

Diagonal argument

\[
\exists f, \quad x < e^x < e^{e^x} < e^{e^{e^x}} < \ldots < f
\]
Three intimately related topics...

(surreal) Numbers

Germs (in HARDY fields)

Transseries
Introduction

Germs
(in HARDY fields)

Transseri
HARDY fields
Let $\mathcal{C}^1$ be the ring of germs at $+\infty$ of continuously differentiable functions $(a, \infty) \to \mathbb{R} \ (a \in \mathbb{R})$.

We denote the germ at $+\infty$ of a function $f$ also by $f$, relying on context.

**Definition**

A **HARDY field** is a subring of $\mathcal{C}^1$ which is a field that contains with each germ of a function $f$ also the germ of its derivative $f'$ (where $f'$ might be defined on a smaller interval than $f$).

**Examples**

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $\mathbb{R}(x,e^x)$, $\mathbb{R}(x,e^x,\log x)$, $\mathbb{R}(x,e^{x^2},\text{erf} \ x)$
HARDY fields capture the somewhat vague notion of functions with “regular growth” at infinity (BOREL, DU BOIS-REYMOND, ...):

Let $H$ be a HARDY field and $f \in H$. Then

$$f \neq 0 \implies \frac{1}{f} \in H \implies \begin{cases} f(x) > 0, \text{ eventually, or} \\ f(x) < 0, \text{ eventually.} \end{cases}$$

Consequently,

- $H$ carries an ordering making $H$ an ordered field:

  $$f > 0 \iff f(x) > 0 \text{ eventually;}$$

- $f$ is eventually monotonic, and

  $$\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm \infty\}.$$
Transseries

(surreal) **Numbers**

**Germs**
(in HARDY fields)

Transseries
The field $\mathbb{T}$ of transseries

$\mathbb{T} := \text{closure of } \mathbb{R} \cup \{x\} \text{ under } \exp, \log \text{ and infinite summation}$

$$e^x + e^{x/2} + e^{x/3} + \cdots - 3e^x + 5(\log x)\pi + 42 + x^{-1} + 2x^{-2} + 6x^{-3} + \cdots + e^{-x}$$
The field $\mathbb{T}$ of transseries

$\mathbb{T} = \mathbb{R}[[\mathcal{M}]] :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$\sum_{m} f_m \cdot m = e^x + e^{x/2} + \cdots - 3e^x + 5(\log x)\pi + 42 + x^{-1} + 2 x^{-2} + 6 x^{-3} + \cdots + e^{-x}$$

$x$: positive infinite indeterminate  \hspace{1cm} $f_m$: coefficient  \hspace{1cm} $m$: transmonomial

$\text{supp } f$: well-based subset of $\mathcal{M}$

disallow $x + \log x + \log \log x + \cdots$ and $e^{-x} + e^{-e^x} + e^{-e^{e^x}} + \cdots$
\( \mathbb{T} \) as an ordered differential field

- With the natural ordering of transseries (via the leading coefficient), \( \mathbb{T} \) is a *real closed ordered field* extension of \( \mathbb{R} \).
- Each \( f \in \mathbb{T} \) can be differentiated term by term (with \( x' = 1 \)):
  \[
  \left( \sum_{n=0}^{\infty} \frac{n! \, e^x}{x^n} \right)' = \sum_{n=0}^{\infty} n! \left( \frac{e^x}{x^n} \right)' = \sum_{n=0}^{\infty} n! \left( \frac{e^x}{x^n} - n \frac{e^x}{x^{n+1}} \right) = \frac{e^x}{x}
  \]
- This yields a *derivation* \( f \mapsto f' \) on the field \( \mathbb{T} \):
  \[
  (f + g)' = f' + g', \quad (f \cdot g)' = f' \cdot g + f \cdot g'
  \]
  Its constant field is \( \{ f \in \mathbb{T} : f' = 0 \} = \mathbb{R} \).
- Given \( f, g \in \mathbb{T} \), the equation \( y' + fy = g \) admits a solution \( y \neq 0 \) in \( \mathbb{T} \).
Surreal numbers

(surreal) Numbers

Germs (in HARDY fields)

Transseries
Class On of ordinal numbers

For any set $L$ of ordinal numbers, there is a smallest ordinal number $\alpha > L$.
Class On of ordinal numbers

For any set $L$ of ordinal numbers, there is a smallest ordinal number $\alpha > L$.

Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a simplest surreal number \{L|R\} such that $L < \{L|R\} < R$. 
Class On of ordinal numbers

For any set $L$ of ordinal numbers, there is a smallest ordinal number $\alpha > L$.

Class No of surreal numbers (CONWAY)

For any sets $L < R$ of surreal numbers, there is a simplest surreal number $\{L|R\}$ such that $L < \{L|R\} < R$.

We have $\text{On} \subseteq \text{No}$ by taking $R = \emptyset$:

\[
\begin{align*}
0 &= \{|\} \\
1 &= \{0|\} \\
2 &= \{0, 1|\} \\
\omega &= \{0, 1, 2, \ldots|\}
\end{align*}
\]
$0 = \{ | \}$
Surreal numbers

\[ 0 = \{|\} \]

\[ -1 = \{|0\} \quad 1 = \{0|\} \]
Surreal numbers

0 = { | }

-1 = { | 0 }

-2 = { | -1, 0 }  \quad -\frac{1}{2} = \{-1 | 0\}  \quad \frac{1}{2} = \{ 0 | 1 \}  \quad 2 = \{ 0, 1 | \}
Surreal numbers

\[0 = \{|\}\]

\[-1 = \{|0\}\]

\[-2 = \{|-1,0\}\]

\[-\frac{1}{2} = \{-1|0\}\]

\[1 = \{0|\}\]

\[2 = \{0,1|\}\]

\[-3\]

\[-1 \frac{1}{2}\]

\[-\frac{3}{4}\]

\[-\frac{1}{4}\]

\[\frac{1}{4}\]

\[\frac{3}{4}\]

\[1 \frac{1}{2}\]

\[3\]
Surreal numbers

$0 = \{|\}$

$-1 = \{|0\}$

$-2 = \{|-1,0\}$

$-3 = \{|...,-1,0\}$

$\omega = \{|...,-1,0\}$

$\frac{1}{2} = \{|-1|0\}$

$\frac{3}{4} = \{|-1|0\}$

$\frac{1}{4} = \{|-1|0\}$

$\frac{1}{2} = \{|0|1\}$

$\frac{1}{4} = \{|0|1\}$

$\frac{3}{4} = \{|0|1\}$

$\frac{1}{2} = \{|0,1|\}$

$2 = \{|0,1|\}$

$3 = \{|0,1|\}$

$\frac{1}{\omega} = \{|0,...,\frac{1}{2},1\}$

$\omega = \{|0,1,...|\}$
Surreal numbers

-1

-2
-3
-4

1

½

0

½

1

2

3

4

-ω

-ω + 1

-ω.2

0

-½

-1

-1½

-1¼

-1¾

-1⅛

-ω

-ω.2

-ω + 1

-ω + 1

0

1

2

3

4

ω

ω + 1

ω + 1
If \( x = \{x^L|x^R\} \) and \( y = \{y^L|y^R\} \), then

\[
x + y := \{x^L + y, x + y^L | x^R + y, x + y^R\}
\]

(Idea: we want \( x^L + y < x + y < x^R + y, \ldots \))
**Definition**

If \(x = \{x_L | x_R\}\) and \(y = \{y_L | y_R\}\), then

\[
x + y := \{x_L + y, x + y_L | x_R + y, x + y_R\}
\]

(Idea: we want \(x_L + y < x + y < x_R + y\), ...)

**Definition**

If \(x = \{x_L | x_R\}\) and \(y = \{y_L | y_R\}\), then

\[
x y := \{xy + x y - x y, \bar{x} y + x \bar{y} - x \bar{y} | xy + x \bar{y} - x \bar{y}, \bar{x} y + x y - \bar{x} y\}
\]

where \(x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R\)
**Arithmetic operations**

**Definition**

If \( x = \{ x^L | x^R \} \) and \( y = \{ y^L | y^R \} \), then

\[
x + y := \{ x^L + y, x + y^L | x^R + y, x + y^R \}
\]

(Idea: we want \( x^L + y < x + y < x^R + y \), ...)

**Definition**

If \( x = \{ x^L | x^R \} \) and \( y = \{ y^L | y^R \} \), then

\[
x y := \{ x y + x y - x y, \tilde{x} y + x \tilde{y} - x \tilde{y} | x y + x \tilde{y} - x \tilde{y}, \tilde{x} y + x \tilde{y} - x \tilde{y} \}
\]

where \( x' \in x_L, x'' \in x_R, y' \in y_L, y'' \in y_R \)

**Theorem (CONWAY)**

No is a real closed field.
In the 1980s, GONSHOR (based on ideas of KRUSKAL) defined an exponential function $\exp: \mathbb{N} \to \mathbb{N}^>0$ that extends $x \mapsto e^x$ on $\mathbb{R}$.

In 2006, BERARDUCCI and MANTOVA (using ideas of VDH and SCHMELING) defined a derivation $\partial_{BM}$ on $\mathbb{N}$ with

$$\ker \partial_{BM} = \mathbb{R}, \quad \partial_{BM}(\omega) = 1, \quad \partial_{BM}(\exp(f)) = \partial_{BM}(f) \cdot \exp(f) \text{ for } f \in \mathbb{N}.$$ 

In a certain technical sense, it is the simplest such derivation that satisfies some natural further conditions.

The BM-derivation on $\mathbb{N}$ behaves in many ways like the derivation on $\mathbb{T}$, with $\omega > \mathbb{R}$ in the role of $x > \mathbb{R}$. For instance, $\partial_{BM}(\log \omega) = \frac{1}{\omega}$. 
Towards a unified theory

- (surreal) Numbers
- Germs (in HARDY fields)
- Transseries
Towards a unified theory

(surreal) Numbers

Germs (in HARDY fields)

Transseries
Towards a unified theory

Numbers (surreal)

H-fields

Germs (in HARDY fields)

Transseries
Towards a unified theory

(surreal)

Numbers

Germs
(in HARDY fields)

H-fields

Hardy
Hausdorff
Hahn

Transseries
Let $K$ be an ordered differential field with constant field $C = \{f \in K: f' = 0\}$.

We define

$f \leq g : \iff |f| \leq c|g|$ for some $c \in C^>0$ \hfill (f is dominated by g)

$f < g : \iff |f| \leq c|g|$ for all $c \in C^>0$ \hfill (f is negligible w.r.t. g)

$f \asymp g : \iff f \leq g \leq f$ \hfill (f is asymptotic to g)

$f \sim g : \iff f - g < g$ \hfill (f is equivalent to g)

Example

In $\mathbb{T}$: $0 < e^{-x} < x^{-10} < 1 < 100 < \log x < x^{1/10} < e^x \sim e^x + x < e^{e^x}$
**Definition**

We call $K$ an **H-field** if

1. $f > C \implies f' > 0$;  
2. $f \asymp 1 \implies f \sim c$ for some $c \in C$.

**Examples**

HARDY fields containing $\mathbb{R}$; ordered differential subfields of $\mathbb{T}$ or $\mathbb{N}_0$ that contain $\mathbb{R}$.

$\mathbb{T}$ admits further elementary properties in addition to being an H-field. It

- has **small derivation**, that is, $f < 1 \implies f' < 1$; and
- is **LIOUVILLE closed**, that is, it is real closed and for all $f, g$, there is some $y \neq 0$ with $y' + fy = g$. 

We view $\mathbb{T}$ model-theoretically as a structure with the primitives

$$0, \ 1, \ +, \ \times, \ \partial \text{ (derivation)}, \ \leq \text{ (ordering)}.$$ 


The elementary theory of $\mathbb{T}$ is completely axiomatized by:

1. $\mathbb{T}$ is a LIOUVILLE closed H-field with small derivation;
2. $\mathbb{T}$ satisfies the intermediate value property for differential polynomials:
   
   Given $P \in \mathbb{T}[Y, Y', \ldots, Y^{(r)}]$ and $u < v$ in $\mathbb{T}$ with $P(u)P(v) < 0$, there exists a $y \in \mathbb{T}$ with $u < y < v$ and $P(y) = 0$

In particular: the theory of $\mathbb{T}$ is decidable.

We also prove a quantifier elimination result for $\mathbb{T}$ in a natural expansion of the above language.
H-field elements as germs

(surreal) Numbers

H-fields

Germs
(in HARDY fields)

Transseries
H-field elements as germs

(surreal) Numbers

Germs (in HARDY fields)

H-fields

Transseries
Theorem (HARDY 1910, BOURBAKI 1951)

Any HARDY field has a smallest LIOUVILLE closed HARDY field extension.
Theorem (HARDY 1910, BOURBAKI 1951)

Any HARDY field has a smallest LIOUVILLE closed HARDY field extension.

Conjecture

Let $H$ be a maximal HARDY field. Then

A. $H$ satisfies the differential intermediate value property.
B. For countable subsets $A < B$ of $H$, there exists an $h \in H$ with $A < h < B$. 
Any HARDY field has a smallest LIOUVILLE closed HARDY field extension.

Let $H$ be a maximal HARDY field. Then

- $H$ satisfies the differential intermediate value property.
- For countable subsets $A < B$ of $H$, there exists an $h \in H$ with $A < h < B$.

$H$ is elementarily equivalent to $\mathbb{T}$ as an ordered differential field.

Under CH, all maximal HARDY fields are isomorphic.
H-field elements as surreal numbers

(surreal) Numbers

H-fields

Germs (in HARDY fields)

Transseries
H-field elements as surreal numbers

(surreal) Numbers

H-fields

Germs
(in Hardy fields)

Transseries
Theorem (JEMS 2019)

Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into $\mathbb{N}_o$. 
Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into $\mathbb{No}$.

Let $\kappa$ be an uncountable cardinal. The field $\mathbb{No}(\kappa)$ of surreal numbers of length $<\kappa$ is an elementary submodel of $\mathbb{No}$. 
**Theorem (JEMS 2019)**

Every $H$-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into $\text{No}$.

**Theorem (JEMS 2019)**

Let $\kappa$ be an uncountable cardinal. The field $\text{No}(\kappa)$ of surreal numbers of length $<\kappa$ is an elementary submodel of $\text{No}$.

**Corollary in progress**

Under CH all maximal HARDY fields are isomorphic to $\text{No}(\omega_1)$.
H-field elements as transseries

(surreal) Numbers

H-fields

Germ (in HARDY fields)

Transseries
H-field elements as transseries

(surreal) Numbers

H-fields

Germ(s) (in HARDY fields)

Transseries
H-field elements as transseries

- Numbers (surreal)
- H-fields
- Germs (in HARDY fields)
- Transseries
Definition (VAN DER HOEVEN 2000, SCHMELING 2001)

A field $\mathbf{T} = \mathbb{R}[[M]]$ with $\text{log}: \mathbf{T}^\rightarrow \rightarrow \mathbf{T}$ is a field of transseries if …

A transserial derivation on $\mathbf{T}$ is a derivation $\partial: \mathbf{T} \rightarrow \mathbf{T}$ such that …
Surreal numbers as transseries

**Definition (Van der Hoeven 2000, Schmeling 2001)**

A field $\mathbb{T} = \mathbb{R}[[M]]$ with $\log: \mathbb{T}^+ \to \mathbb{T}$ is a field of transseries if …

A transserial derivation on $\mathbb{T}$ is a derivation $\partial: \mathbb{T} \to \mathbb{T}$ such that …

**Theorem (Berarducci–Mantova, 2015)**

$\mathbb{N}$ is a field of transseries and $\partial_{BM}$ is a transserial derivation.
Surreal numbers as transseries

**Definition (Van der Hoeven 2000, Schmeling 2001)**

A field \( T = \mathbb{R}[[M]] \) with \( \log: T^> \rightarrow T \) is a **field of transseries** if …

A **transserial derivation** on \( T \) is a derivation \( \partial: T \rightarrow T \) such that …

**Theorem (Berarducci–Mantova, 2015)**

\( \text{No} \) is a field of transseries and \( \partial_{\text{BM}} \) is a transserial derivation.

**Corollary**

Any H-field with constant field \( \mathbb{R} \) can be embedded in a field of transseries with a transserial derivation.
What next?

(surreal) Numbers

H-fields

Transseries

Gerns
(in HARDY fields)
What next?

(surreal) Numbers

Germs (in HARDY fields)

beyond H-fields

Transseries
What next?

(surreal) **Numbers**

= beyond H-fields

**Germ**

(in HARDY fields)

**Transseries**
Équations d'itération

\[ \exp_\omega(x + 1) = \exp \exp_\omega x \]

→ Croissance plus rapide que \( e^x, e^{e^x}, e^{e^{e^x}}, \ldots \)

→ Kneser 1950 : il existe une solution réelle analytique \( \exp_\omega \)
Iterated exponentials and logarithms

\[ \exp_\omega (x + 1) = \exp \exp_\omega x \]
\[ \exp_{\omega^2} (x + 1) = \exp_\omega \exp_{\omega^2} x \]

\[ \vdots \]

→ stronger growth than \( e^x, e^{e^x}, \ldots, \exp_\omega x, e^{\exp_\omega x}, \ldots, \exp_\omega \exp_\omega x, \ldots \)
Transseries not completely closed...

Iterated exponentials and logarithms

\[
\begin{align*}
\exp_\omega (x + 1) &= \exp \exp_\omega x \\
\exp_{\omega^2} (x + 1) &= \exp_\omega \exp_{\omega^2} x \\
\end{align*}
\]

→ stronger growth that \( e^x, e^{e^x}, \ldots, \exp_\omega x, e^{\exp_\omega x}, \ldots, \exp_\omega \exp_\omega x, \ldots \)

Functional equations

\[
f(x) = \sqrt{x} + e^{f(\log x)} = \sqrt{x} + e^{\sqrt{\log x} + e^{\sqrt{\log\log x} + \ldots}}
\]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega(x + 1) &= \exp \exp_\omega x \\
\exp_{\omega^2}(x + 1) &= \exp_\omega \exp_{\omega^2} x \\
\vdots \\
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
\vdots
\end{align*}
\]
Hyperlogarithms and hyperexponentials

\[ \exp_\omega(x + 1) = \exp \exp_\omega x \]
\[ \exp_{\omega^2}(x + 1) = \exp_\omega \exp_{\omega^2} x \]
\[ \vdots \]
\[ \log_\omega \log x = \log_\omega x - 1 \]
\[ \log_{\omega^2} \log_\omega x = \log_{\omega^2} x - 1 \]
\[ \log_\omega x = \frac{1}{x \log x \log \log x \ldots} \]
\[ \log_\alpha x = \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x} \]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega(x + 1) &= \exp \exp_\omega x \\
\exp_{\omega^2}(x + 1) &= \exp_\omega \exp_{\omega^2} x \\
&\vdots \\
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
&\vdots \\
\log_\omega x &= \frac{1}{x \log x \log \log x \ldots} \\
\log_\alpha x &= \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x}
\end{align*}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[ f_0(x) \]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega(x + 1) &= \exp \exp_\omega x \\
\exp_{\omega^2}(x + 1) &= \exp_\omega \exp_{\omega^2} x \\
&\vdots
\end{align*}
\]

\[
\begin{align*}
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
&\vdots
\end{align*}
\]

\[
\log_\omega x = \frac{1}{x \log x \log \log x \ldots}
\]

\[
\log_\alpha x = \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[
f_{-1}(x) < f_0(x) < f_1(x)
\]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega(x+1) &= \exp \exp_\omega x \\
\exp_\omega^2(x+1) &= \exp_\omega \exp_\omega^2 x \\
&\vdots \\
\log_\omega \log x &= \log_\omega x - 1 \\
\log_\omega^2 \log_\omega x &= \log_\omega^2 x - 1 \\
&\vdots \\
\log_\omega x &= \frac{1}{x \log x \log \log x \cdots} \\
\log_\alpha x &= \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x}
\end{align*}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[
\begin{align*}
f_{-2}(x) &< f_{-1}(x) < f_{-1/2}(x) < f_0(x) < f_{1/2}(x) < f_1(x) < f_2(x)
\end{align*}
\]
Hyperlogarithms and hyperexponentials

\[
\begin{align*}
\exp_\omega (x+1) &= \exp \exp_\omega x \\
\exp_{\omega^2} (x+1) &= \exp_\omega \exp_{\omega^2} x \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\log_\omega \log x &= \log_\omega x - 1 \\
\log_{\omega^2} \log_\omega x &= \log_{\omega^2} x - 1 \\
\vdots
\end{align*}
\]

\[
\log_\omega x = \frac{1}{x \log x \log \log x \ldots}
\]

\[
\log_\alpha x = \int \prod_{\beta < \alpha} \frac{1}{\log_\beta x}
\]

Nested hyperseries

Solutions de \( f(x) = \sqrt{x} + e^{f(\log x)} \):

\[
\begin{align*}
\cdots < f_{-2}(x) < \cdots < f_{-1}(x) < \cdots < f_0(x) < \cdots < f_{1/2}(x) < \cdots < f_1(x) < \cdots < f_2(x) < \cdots
\end{align*}
\]
Conjecture (vdH 2006)

For an appropriate definition of the class $\text{Hy}$ of hyperseries, we have $\mathbb{N}_0 \cong \text{Hy}$ for the map $\phi: \text{Hy} \rightarrow \mathbb{N}_0; f \mapsto f(\omega)$. 
Conjecture (vdH 2006)

For an appropriate definition of the class $\text{Hy}$ of hyperseries, we have $\mathbb{N}_0 \cong \text{Hy}$ for
the map $\phi: \text{Hy} \rightarrow \mathbb{N}_0; f \mapsto f(\omega)$.

Proof. By constructing a Conway bracket $\{\} \}$ on $\text{Hy}$. 

Conjecture (vdH 2006)

For an appropriate definition of the class $\text{Hy}$ of hyperseries, we have $\mathbb{N}_0 \cong \text{Hy}$ for the map $\phi: \text{Hy} \rightarrow \mathbb{N}_0; f \mapsto f(\omega)$.

Proof. By constructing a Conway bracket $\{\}$ on $\text{Hy}$.

Examples:

$\{x, e^x, e^{e^x}, \ldots\} = \exp_\omega x$

$\{\sqrt{x}, \sqrt{x} + e^{\sqrt{\log x}}, \ldots, \sqrt{x} + e^{2\sqrt{\log x}}, 2\sqrt{x}\} = f_0(x)$

$\{x^2, e^{\log^2 x}, e^{e^{\log^2 \log x}}, \ldots, e^{e^{e^{\sqrt{\log \log x}}}}, e^{e^{\sqrt{\log x}}}, e^{\sqrt{x}}\} = \exp_\omega \left(\log_\omega x + \frac{1}{2}\right)$
Thank you!

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