Maurice’s Siblings

Many years of wonderful collaboration

Claude Laflamme
University of Calgary

Collaborators:
N. Sauer and R. Woodrow, University of Calgary
**Definition (Siblings)**

- Given two structures $\mathcal{A}$ and $\mathcal{B}$, write:

  $\mathcal{A} \leq \mathcal{B}$ if there is a monomorphism (embedding) from $\mathcal{A}$ to $\mathcal{B}$,
  $\mathcal{A} \equiv \mathcal{B}$ if both $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$.

In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are *siblings*, or *equimorphic* (or even “twins” if they are non-isomorphic siblings).

- $\text{sib}(\mathcal{A})$ denotes the number of siblings of $\mathcal{A}$, up to isomorphism.
Example

1. $\text{sib}(A) = 1$: The following structures have a single sibling.
   - Sets (Cantor-Schröder-Bernstein).
   - Vector spaces over a fixed field.
   - Finitely generated abelian group.
   - Uncountable algebraically closed field.

2. $\text{sib}(A) = \aleph_0$: A ray (as a graph) has countably (infinite) siblings

3. $\text{sib}(A) = 2^{\aleph_0}$: The rationals have continuum siblings.

\[
\sum_{i \in \omega} \mathbb{Z} \chi_X(i)
\]
Conjecture (Thomassé - circa 2000)

*If \( A \) is a countable relational structure, then \( \text{sib}(A) = 1, \aleph_0, \) or \( 2^{\aleph_0} \).*

Conjecture (Bonato - Tardif 06)

*Any tree \( T \) has either infinitely many twins, or none. That is \( \text{sib}(T) = 1 \) or \( \text{sib}(T) \geq \aleph_0 \) (Tree Alternative Conjecture).*

Conjecture (Tyomkim 09)

*\( \text{sib}(T) \geq \aleph_0 \) for any locally finite tree \( T \) which has a non-surjective embedding, except for a ray.*

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In the category of connected graphs with loops, the following structure has exactly 2 siblings:

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· · ·
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The only other (connected) sibling (up to isomorphism) is:

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· · ·
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As a relational structure, we find infinitely many siblings:

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· · ·
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Similarly in the category of *connected posets*, the one way infinite fence has exactly two siblings:

![Diagram of a one way infinite fence with two siblings]

And again as a relational structure, the above structure has infinitely many siblings.

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Thomassé’s conjecture is open even for countable simple graph: there is no known example of a countable simple graph $G$ where $\text{sib}(G)$ is not $1$, $\aleph_0$, or $2^{\aleph_0}$. 
I: Back to Trees

Theorem (Tits 70)

Every automorphism of a tree preserves a vertex (a rotation), an edge (an inversion), or a two-way infinite path (a translation).

Theorem (Halin 73)

Let $f$ be an embedding of a tree $T$ into itself. Then either there is:

1. A fixed vertex; or
2. An edge reversed by $f$; or
3. A two-way infinite path preserved by $f$ (and not the previous cases); or
4. A ray $C$ preserved by $f$ (and a vertex not in the range of $f$).

Furthermore, each case excludes the others.

Theorem (Halin 90)

Every rayless tree has a fixed vertex or a fixed edge which is preserved by every self-embedding.
Theorem (Bonato - Tardif 06)

The tree alternative conjecture holds for rayless trees.

Theorem (Tyomkin 09)

The tree alternative conjecture holds for rooted trees.

Theorem (Bonato - Bruhn - Diestel - Sprüssel 11)

The (graph) alternative conjecture holds for rayless graphs.
Definition (Scattered Tree)
A tree is scattered if it does not embed a subdivision of the binary tree.

Definition (End of a Tree)
An end of a tree is an equivalence class of “almost equal” rays.

Fact (Jung 69 - Polat 96)
A tree is scattered iff the space of ends is topologically scattered.

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Theorem (LPS ~15)

The tree alternative conjecture holds for scattered trees, and Tyomkyn’s conjecture holds for locally finite scattered trees.

Theorem (LPS ~15)

*If a tree is scattered, then either there is one vertex, one edge, or a set of at most two ends preserved by every embedding.*

**Definition**

- An end $e$ is preserved *forward* (resp. *backward*) by an embedding $f$ if there is some ray $C \in e$ such that $f[C] \subseteq C$ (resp. $C \subseteq f[C]$).
- An end $e$ is *almost rigid* if it is preserved backward and forward by every embedding.
- A ray $C = \{x_0, \ldots, x_n, \ldots\}$ is *regular* if the number of pairwise non-equimorphic rooted trees $T_{x_i}$ is finite (and an end is *regular* if it contains some regular ray).
Theorem (LPS ~15)

(i) If a scattered $T$ does not contain a vertex or an edge preserved by every embedding, or an almost rigid end, and has a non-surjective embedding, then $\text{sib}(T) = \infty$ unless $T$ is the one-way infinite path.

(ii) If $T$ has an almost rigid end, then $\text{sib}(T) = 1$ if and only if $\text{sib}(T(\rightarrow x)) = 1$ for every vertex $x$, otherwise $\text{sib}(T) = \infty$.

(iii) If $T$ has a non-regular and not almost rigid end preserved forward by every embedding then $\text{sib}(T) \geq 2^{\aleph_0}$.

Corollary (LPS ~16)

Let $T$ be a scattered tree with $\text{sib}(T) < 2^{\aleph_0}$. Then there exists a vertex or an edge or a two-way infinite path or a one-way infinite path or an almost rigid end preserved by every embedding of $T$. 
**Theorem (Hamann ~16)**

Let $G$ be a monoid of embeddings of a tree $T$. Then either:

1. *There is a vertex, an edge or a set of at most two ends preserved by each member of $G$;*

2. *Or $G$ contains a submonoid freely generated by two embeddings.*
Problem (How many siblings?)
II: Relational Structures – The Case of Chains

Theorem (LPW ~14)

- Thomassé’s Conjecture holds for (countable) chains, that is $\text{sib}(C) = 1, \aleph_0, \text{ or } 2^{\aleph_0}$,
- The (chain) alternative conjecture holds for all chains, that is $\text{sib}(C) = 1$ or $\text{sib}(C) = \infty$.

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Example

- \( sib(C) = 1 \) for any finite sum of ordinals or reverse ordinals.
- \( sib(\omega^* \cdot \omega) = \aleph_0 \) (and \( sib(\lambda^* \cdot \omega) = |\lambda| \)).

\[
\begin{array}{ccccccc}
\omega^* \cdot \omega & & & & & & \\
\downarrow & & & & & & \\
\omega^* & \omega^* & \omega^* & & & & \\
\downarrow & & & & & & \\
\mathcal{P} & \omega^* & \omega^* & \omega^* & & & \\
\end{array}
\]

- \( sib(\mathcal{Q}) = 2^{\aleph_0} \).

Corollary: All non-scattered countable chains have \( 2^{\aleph_0} \) siblings.

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Definition (Surordinal - Slater & Jullien)

A chain $C$ is a **surordinal** if $1 + \omega^*$ does not embed in $C$.

Theorem (Scattered Chains with Few Siblings)

Let $C$ be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1. $\text{sib}(C) = \kappa$ and $C$ is scattered;
2. $\kappa = 1$, or $\kappa \geq \aleph_0$ and $C$ is a finite sum of surordinals and of reverse of surordinals, and if $C = \sum_{j<m} D_j$ is such a sum with $m$ minimum then $\max\{\text{sib}(D_j) : j < m\} = \kappa$.

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P. Jullien, Contribution à l’étude des types d’ordres dispersés, Thèse Doctorat d’État, Université de Marseille, 27 juin 1968.
Theorem (Chains with Few Siblings)

Let $C$ be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1. $sib(C) = \kappa$.
2. $C = \sum_{i \in D} C_i$, where:
   - $D$ is dense (singleton or infinite),
   - each $C_i$ is scattered, $sib(C_i) = 1$ for all but finitely many $i \in D$,
   - $\max\{sib(C_i) : i \in D\} = \kappa$, and
   - every embedding $f : C \to C$ preserves each $C_i$.

Corollary (Alternative Conjecture for Chains – LPW *14)

$sib(C) = 1$ or $sib(C) \geq \aleph_0$ for any chain $C$. 

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Hausdorff’s condensation and rank arguments.
Example (Dushnik and Miller)
It is possible to have $C = \sum_{i \in \mathbb{R}} C_i$, and $\text{sib}(C) = 1$.

Problem

Suppose that $C = \sum_{i \in D} C_i$, where:

- $D$ is embedding rigid,
- each $C_i$ is scattered, $\text{sib}(C_i) = 1$ for all but finitely many $i \in D$, and
- $\max\{\text{sib}(C_i) : i \in D\} = \kappa$.

Does it follow that $\text{sib}(C) = \kappa$?

Problem

What about partial orders?

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III: $\aleph_0$-categorical Relational Structure

**Definition**

*R* is *finitely partitionable* if there is a partition of the domain *E* of *R* into finitely many sets such that every permutation of *E* which preserves each block of the partition is an automorphism of *R*.

**Theorem (Hodkinson & Macpherson 88)**

A countable structure *R* in a finite language, or even infinite language if *R* is $\aleph_0$-categorical, is such that every structure with the same finite substructures is isomorphic to *R* if and only if *R* is finitely partitionable.

**Theorem (LPSW 19)**

- The alternative conjecture holds for any countable $\aleph_0$-categorical relational structure.
- Furthermore, $\text{sib}(R) = 1$ if and only if *R* is finitely partitionable.
**Definition (Monomorphic Decomposition)**

- Let $R$ a relational structure on a set $E$. A subset $E'$ of $E$ is a *monomorphic part* of $R$ if for every integer $k$ and every pair $A, A'$ of $k$-element subsets of $E$, the induced structures on $A$ and $A'$ are isomorphic whenever $A \backslash E' = A' \backslash E'$.

- A *monomorphic decomposition* of $R$ is a partition $\mathcal{P}$ of $E$ into monomorphic parts.

- A monomorphic part which is maximal for inclusion is a *monomorphic component* of $R$.

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The monomorphic components of $R$ form a monomorphic decomposition of $R$ of which every monomorphic decomposition of $R$ is a refinement.
Remark

We may assume $R$ is prehomogeneous.

Theorem (LPSW 19)

- **Case 1:** If $R$ has a finite decomposition, then $\text{sib}(R) = 1$ or $\text{sib}(R) = 2^{\aleph_0}$.

  If one of the infinite component is not strongly indiscernible, then $\text{sib}(R) = 2^{\aleph_0}$; $\text{sib}(R) = 1$ otherwise.

- **Case 2:** If $R$ has no finite decomposition, then $\text{sib}(R) = \infty$.

  If $R$ has infinitely many infinite monomorphic components, then $\text{sib}(R) = 2^{\aleph_0}$.

Maurice + Frasnay + Fraïssé

Chainable structures, bichains, indicative sequences.
**Problem**

If $R$ is a countable $\aleph_0$-categorical relational structure, then

$$\text{sib}(A) = 1, \aleph_0, \text{ or } 2^{\aleph_0}.$$  

Moreover $\text{sib}(R) \leq \aleph_0$ if and only if $R$ is cellular (Schmerl).

**Remark**

$R$ is a countable $\aleph_0$-categorical relational structure iff $\text{Aut}(R)$ is oligomorphic.

Thus if a group $G$ acts on a set $E$, call its closure $\overline{G}$ the set of $G$-embeddings, and define a $G$-copy to be the image of $E$ under some $G$-embedding, and finally a $G$-sibling is a subset of $E$ containing a $G$-copy.

**Theorem (LPSW 19)**

The alternative conjecture holds for any closed oligomorphic group $G$ acting on a countable set $E$. 

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C. Laflamme 23 / 24
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