F$_3$-reconstruction

Youssef BOUDABBOUS and Christian DELHOMMÉ

Université de la Réunion

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The formalism of reversible labeled 2-structures (EHR)

Labeled 2-structures

Binary relational structures as (reversible) labeled 2-structures.

We consider a set \( \Lambda \), of labels, endowed with an involution \( \lambda \mapsto \lambda^* \) (\( \lambda^{**} = \lambda \)).

A label \( \lambda \in \Lambda \) is selfdual if \( \lambda^* = \lambda \).

We denote by \( \overrightarrow{\Lambda} \subseteq \Lambda \) the set of non-selfdual labels.

**Definition ((reversible) Labeled 2-structure (EHR))**

A (reversible) \( \Lambda \)-2-structure (\( \Lambda \)2s), on \( V \) (of vertex set \( V \)), is a mapping \( \mathfrak{A} : V^2_* \to \Lambda \) defined on the set of ordered pairs of distinct vertices, satisfying \( \mathfrak{A}(y, x) = (\mathfrak{A}(x, y))^* \).

We write \( x \xrightarrow{\lambda} y \) (or just \( x \xrightarrow{\lambda} y \)) for \( \mathfrak{A}(x, y) = \lambda \). Thus \( x \xrightarrow{\lambda} y \iff y \xrightarrow{\lambda^*} x \).

Example : The class of usual directed graphs (digraphs) corresponds to \( \Lambda \) with two selfdual labels 0 and 1 and two non-selfdual labels + and − = +*.

- Simple graphs correspond to 2-structures with labels in \{0, 1\}.
- Tournaments correspond to 2-structures with labels in \{+, −\}.
- Complete and edge-free simple graphs correspond to the constant 2-structures.
Fraïssé’s reconstruction from small restrictions

$F_k$-Reconstruction

The $\Lambda 2s \mathcal{A} \upharpoonright W$ induced on a set $W$ of vertices of a $\Lambda 2s \mathcal{A}$:

$$\forall (x, y) \in W^2 : (\mathcal{A} \upharpoonright W)(x, y) = \mathcal{A}(x, y)$$

An isomorphism from $\mathcal{A}$, on $V$, to $\mathcal{A}'$ on $V'$, is a bijection $f : V \rightarrow V'$ such that:

$$\forall (x, y) \in V^2 : \mathcal{A}'(f(x), f(y)) = \mathcal{A}(x, y)$$

$\mathcal{A}$ and $\mathcal{A}'$ are isomorphic if there is such an isomorphism, written $\mathcal{A} \simeq \mathcal{A}'$.

**Definition** ($k$ : a positive integer.)

Two $\Lambda 2s \mathcal{A}$ and $\mathcal{A}'$ with the same vertex set $V$ are ($\leq k$)-hypomorphic, written $\mathcal{A} \trianglelefteq_k \mathcal{A}'$ if their restrictions to each set of at most $k$ vertices are isomorphic:

$$\forall W \in \binom{V}{\leq k} : \mathcal{A} \upharpoonright W \simeq \mathcal{A}' \upharpoonright W$$

We also say that $\mathcal{A}'$ is an $F_3$-binom of $\mathcal{A}$.

**Definition**

A $\Lambda 2s \mathcal{A}$ is ($\leq k$)-reconstructible if every $\Lambda 2s$ ($\leq k$)-hypomorphic with $\mathcal{A}$ is isomorphic to $\mathcal{A}$:

$$\mathcal{A} \trianglelefteq_k \mathcal{A}' \implies \mathcal{A} \simeq \mathcal{A}'$$
Lopez’ reconstruction theorem

F$_6$-Reconstructibility theorem of finite digraphs

A Λ2s $\mathcal{A}$ is $(\leq k)$-reconstructible if $\mathcal{A}^k \cong \mathcal{A}' \implies \mathcal{A} \cong \mathcal{A}'$.

Theorem (G. Lopez (1972))

Every finite digraph is $(\leq 6)$-reconstructible.

Characterisations of $(\leq k)$-reconstructibility:

- Y. Boudabbous (2002): $k = 5$ for finite digraphs
- Y. Boudabbous and G. Lopez (2005): $k \in \{3, 4\}$ for finite digraphs
- Y. Boudabbous and C. Delhommé (2014): $k \in \{4, 5, 6\}$ for possibly infinite digraphs.

Here: $k = 3$ for short Λ2s.

We characterise the F$_3$-binoms of a robust Λ2s, and infer a characterisation of substitutions preserving $(\leq 3)$-reconstructibility, yielding a characterisation of $(\leq 3)$-reconstructibility for short Λ2s.
Modular partition. Substitution

A **modular partition** of a $\Lambda 2s \mathcal{A}$ is any partition $\mathcal{P}$ of its vertex set $V$ such that:

$$\forall (X, Y) \in \mathcal{P}^2 : (\exists \lambda \in \Lambda \forall (x, y) \in X \times Y : x \xrightarrow{\lambda} y)$$

The **quotient** $\Lambda 2s \mathcal{A}/\mathcal{P}$, on $\mathcal{P}$:

$$\forall (X, Y) \in \mathcal{P}^2 \forall (x, y) \in X \times Y : (\mathcal{A}/\mathcal{P})(X, Y) = \mathcal{A}(x, y)$$

Example: partition of a linear order into intervals.

$\mathcal{A}$ is recoverable from $\mathcal{A}/\mathcal{P}$ and $(\mathcal{A} | M : M \in \mathcal{P})$ as:

$$\mathcal{A} = (\mathcal{A}/\mathcal{P})[\mathcal{A} | M : M \in \mathcal{P}]$$

**Substitution**: In general, for a $\Lambda 2s \mathcal{B}$ on $V$ and a family of $\Lambda 2s (\mathcal{A}_v : v \in V)$ with pairwise disjoint vertex sets $V_v$, one defines $\mathcal{A} = \mathcal{B}[\mathcal{A}_v : v \in V]$ on $\cup\{V_v : v \in V\}$ (substitution **along** $\mathcal{B}$):

$$\forall (x, y) \in V_v \times V_w, x \neq y \Rightarrow \begin{cases} v = w \Rightarrow \mathcal{A}(x, y) = \mathcal{A}_v(x, y) \\ v \neq w \Rightarrow \mathcal{A}(x, y) = \mathcal{B}(v, w) \end{cases}$$

**Trivial** modular partitions: **coarse** and **discrete** modular partitions.

$\mathcal{A}$ **indecomposable**: has only trivial modular partitions.
Given \( \Lambda \) 2s \( \mathcal{A} \) on \( V \), for \( M \subseteq V \) TFAE:

- \( M \) is a member of a modular partition of \( \mathcal{A} \).
- \( \{M\} \cup \{\{x\} : x \in V \setminus M\} \) is a modular partition of \( \mathcal{A} \).
- \( M \neq \emptyset \) and is undistinguishable from the outside:

  \[ \forall x \in V \setminus M \forall y, y' \in M : (x \xrightarrow{\lambda} y \iff x \xrightarrow{\lambda} y') \quad (1) \]

**Definition**

A (possibly empty) set \( M \) of vertices satisfying (1) is called a module of \( \mathcal{A} \).

**Example:** Intervals of a linear order. A partition of \( V \) is modular if and only if all its members are modules. **Trivial** modules: \( \emptyset \), \( V \), singletons. \( \mathcal{A} \) is **indecomposable** if and only if its modules are trivial. The \( \Lambda \) 2s with at most two vertices are trivially indecomposable. \( \mathcal{A} \) is **prime** if it is indecomposable with more than two vertices.

**Example**

- The minimum prime poset (resp. simple graph) \( N \), w.r.t. embeddability.
Basic and short $\Lambda 2s$

Basic and robust $\Lambda 2s$

A $\Lambda 2s$ $\mathfrak{A}$ is **prime** if it is indecomposable with more than two vertices. Note that a $\Lambda 2s$ has at most one prime quotient.

$\mathfrak{A}$ is **$\lambda$-linear** ($\lambda \in \overrightarrow{\Lambda}$) if its vertex set is strictly linearly ordered by $x \xrightarrow{\lambda} y$.

**Definition (Basic $\Lambda 2s$)**

A **nonempty** $\Lambda 2s$ is **basic** if it is **prime** or **linear** or **constant**.

**Lemma**

*Each finite $\Lambda 2s$ $\mathfrak{A}$ with at least two vertices admits a noncoarse basic quotient.*

**Corollary**

*The class of nonempty finite $\Lambda 2s$ is the closure of the class of singleton $\Lambda 2s$ under substitution along finite basic $\Lambda 2s$.*

**Definition (Short $\Lambda 2s$)**

The class of nonempty **short** $\Lambda 2s$ is the closure of the class of singleton $\Lambda 2s$ under substitution along basic $\Lambda 2s$. 
Basic substitutions preserving ($\leq 3$)-reconstructibility

**Definition (Short $\Lambda 2$s)**

The class of nonempty short $\Lambda 2$s is the closure of the class of singleton $\Lambda 2$s under substitution along basic $\Lambda 2$s.

We characterise these substitutions along basic $\Lambda 2$s that preserve ($\leq 3$)-reconstructibility. Call them $F_3$-specific.

**Theorem**

*The class of nonempty short $F_3$-reconstructible $\Lambda 2$s is the closure of the class of singleton $\Lambda 2$s under $F_3$-specific substitutions.*
**Lemma**

*Each nonempty \( \Lambda 2s \ A \) admits a finest basic quotient.*

If \( A \) is finite with at least two vertices then this quotient is noncoarse.

**Definition**

A \( \Lambda 2s \) is **robust** if it has a single vertex or a noncoarse basic quotient. In this case, the classes of its finest basic quotient are the **components** of \( A \). They form its **canonical modular partition** and the corresponding (basic) quotient is the **canonical quotient** of \( A \) (or its frame).

**Proposition (Cf. Gallai, Kelly, EHR, ...)**

*If a \( \Lambda 2s \ A \) has a noncoarse basic quotient then the classes of the finest one are the maximal proper strong modules of \( A \). In other words the components of a nonsingleton robust \( \Lambda 2s \) are its maximal proper strong modules.*

**Definition**

A module of \( A \) is **strong** if it is comparable with each module that it meets.

Two meeting strong modules are comparable. Trivial modules are strong.
Robust modules

- For any two distinct vertices $x$ and $y$ of $\mathcal{A}$, the structure induced on the least strong module $R$ containing them both is robust, and these vertices lie in two distinct components of $\mathcal{A} \upharpoonright R$.
- A strong module is join-reducible in the lattice of strong modules if and only if it is the join of two singletons.
- The components of a join-reducible strong module $R$ are the maximal strong modules properly included in $R$.

**DEFINITION :** A robust module of a $\Lambda 2s$ is singleton or a join-reducible element of the lattice of strong modules.

- A strong module $R$ of $\mathcal{A}$ is robust iff the induced structure $\mathcal{A} \upharpoonright R$ is robust.
- A nonempty $\Lambda 2s$ is robust if and only if its vertex set is a robust module.
- The collection of robust modules of $\mathcal{A}$ is a generalised tree underlying a term of basic substitutions evaluating to this structure: to each inner node $R$ is associated its frame. For any two distinct vertices $x$ and $y$, $\mathcal{A}(x, y) = \mathcal{A}_R(X, Y)$ where $R$ is the least strong module containing $x$ and $y$ and $X$ and $Y$ are its components containing respectively $x$ and $y$.

- A robust module has kind prime, linear or constant, according to its frame.
- A robust module of kind linear or constant has a colour: $\hat{\lambda} = \{\lambda, \lambda^*\}$ if the frame is a $\lambda$-chain; $\hat{\lambda} = \{\lambda\}$ if the frame is constant of value $\lambda$. 
An arc of a $\Lambda$2s $\mathcal{A}$ is an ordered pair $(x, y) \in V^2_*$, of distinct vertices, such that $\mathcal{A}(x, y) \in \overrightarrow{\Lambda}$ (i.e. $\mathcal{A}(y, x) \neq \mathcal{A}(x, y)$).

The dual $\mathcal{A}^*$ of a $\mathcal{A}$ is the structure with the same vertices obtained by reversing the arcs:

$$x \xrightarrow{\lambda} \mathcal{A}^* y \iff y \xrightarrow{\lambda} x \iff x \xrightarrow{\lambda^*} \mathcal{A} y$$

A 2-structure is arc-connected if the simple graph defined on its vertex set by $\mathcal{A}(x, y) \in \overrightarrow{\Lambda}$ is connected.

Given a non-selfdual label $\lambda \in \overrightarrow{\Lambda}$, a $\lambda$-tournament is a $\Lambda$2s of range included in $\{\lambda, \lambda^*\}$.

Thus linear $\Lambda$2s are particular tournaments.

The colour of a label $\lambda \in \Lambda$ is $\hat{\lambda} := \{\lambda, \lambda^*\}$. The colour of an ordered pair of distinct vertices of a $\Lambda$2s is the colour of its label.

A 3-vertex $\Lambda$2s is a peak if it is of the form:

A flag is a 3-vertex $\Lambda$2s of which the three unordered pairs have distinct colours.
Robust structures of constant kind

Lemma (F₃-Reconstruction of robust structures of constant kind)

Consider a robust Λ-2-structure Λ of constant kind.

1. Each F₃-binom of Λ is robust of constant kind with the same frame.
2. Λ is F₃-reconstructible if and only if each of its modular components is so.
Robust structures of linear kind

**PROPOSITION:** Consider a robust $\mathcal{A}$-2-structure $\mathcal{A}$ of $\lambda$-linear kind. Thus the binary relation $(\mathcal{A}/\mathcal{P})(M, N) = \lambda$ is a strict linear ordering $<$ on the set $\mathcal{P}$ of modular components of $\mathcal{A}$. Consider the partition of $\mathcal{P}$ of which the classes are the singletons $\{M\}$ for $M$ a modular component failing to be a $\lambda$-tournament, and the maximal $<$-intervals containing only $\lambda$-tournament modular components. Then denote by $Q$ the partition of the vertex set $V$ of $\mathcal{A}$ lifting this partition of $\mathcal{P}$. Thus the members of $Q$ are the modular components that are not $\lambda$-tournaments and the maximal $\lambda$-tournament modules that are not properly included in a component.

1. For a structure $\mathcal{A}'$ on $V$ to be an $F_3$-binom of $\mathcal{A}$ it is necessary and sufficient that $Q$ be a modular partition of $\mathcal{A}'$, $\mathcal{A}'/Q = \mathcal{A}/Q$, and for each $M \in Q$, $\mathcal{A}' \upharpoonright M$ be an $F_3$-binom of $\mathcal{A} \upharpoonright M$.

2. Each $F_3$-binom of $\mathcal{A}$ is robust of linear kind of colour $\lambda$, with the same modular components as $\mathcal{A}$ (but it may have a different frame).

3. The structure $\mathcal{A}$ is $F_3$-reconstructible if and only if for each $M \in Q$, the restriction $\mathcal{A} \upharpoonright M$ is $F_3$-reconstructible. (There is choice in both directions: in the proof of the sufficiency to paste isomorphisms, and in the proof of the necessity to deform some particular modules.)

4. A $\lambda$-tournament member of $Q$ is $F_3$-reconstructible iff it is the union of finitely many pairwise isomorphic and $F_3$-reconstructible components of $\mathcal{A}$. 
Corollary

A robust structure of $\lambda$-linear kind is $F_3$-reconstructible if and only if each of its modular components is $F_3$-reconstructible, and each maximal interval of its frame on which the components are $\lambda$-tournaments is finite and these finitely many components are pairwise isomorphic.
Robust structures of prime kind

-The **dual** of a 2-structure $\mathcal{A}$ is the structure $\mathcal{A}^*$ with the same vertex set given by $\mathcal{A}^*(x, y) = \mathcal{A}(y, x)$.

-$\mathcal{A}$ is **selfdual** if it is isomorphic to $\mathcal{A}^*$.

-The **neutral uniformisation** of a 2-structure $\mathcal{A}$ is the structure $\tilde{\mathcal{A}}$ obtained from $\mathcal{A}$ by identifying all selfdual labels.

-If this neutral uniformisation $\tilde{\mathcal{A}}$ has a prime quotient then the structure obtained from it by reversing all arcs transverse to its components is the **pseudodual** of $\mathcal{A}$.

**Lemma**

*If a $\Lambda 2s$ $\mathcal{A}$ is arc-connected peak-free and robust of prime kind then so is $\tilde{\mathcal{A}}$.***
**EXAMPLE** : An arc-connected prime digraph $\mathcal{A}$ for which the oriented graph $\tilde{\mathcal{A}}$ fails to be robust.

As for the digraph $\mathcal{A}$, on the right, the dotted lines are non-edges, the dashed lines are symmetric edges, and the mixed line between a vertex and a set of vertices indicates that every pair between this vertex and each vertex inside this set is a non-edge or a symmetric edge (here the line is drawn vertically, the vertex is $3n + 1$ and the set of vertices is $R_n$). The type of adjacency of the pairs of vertices between which no arc nor line is drawn is actually irrelevant for obtaining a prime digraph of which the neutral uniformisation fails to be robust.
**Definition (F₃-special structure)**

A Λ-2-structure $\mathfrak{A}$ is $\lambda$-special, for a label $\lambda \in \Lambda$, if it satisfies the two points below. Say that $\mathfrak{A}$ is special if it is $\lambda$-special for some $\lambda$.

1. $\mathfrak{A}$ is arc-connected, peak-free and admits a prime quotient.
   In this case, its neutral uniformisation $\tilde{\mathfrak{A}}$ too has these three properties.
2. Each component of $\tilde{\mathfrak{A}}$ is a $\lambda$-tournament and no arc of any flag is transverse to such components.
PROPOSITION: Consider a robust reversible $\Lambda$-2-structure $\mathcal{A}$ of prime kind. Let $V$ denote its vertex set and $\mathcal{P}$ its canonical modular partition.

A structure $\mathcal{A}'$ on $V$ is an $F_3$-binom of $\mathcal{A}$ if and only if $\mathcal{P}$ is a modular partition of $\mathcal{A}'$, for each modular component the corresponding restrictions of $\mathcal{A}$ and $\mathcal{A}'$ are $F_3$-hypomorphic, and:

1. either Frame $\mathcal{A}' = \mathcal{A}'/\mathcal{P} = \mathcal{A}/\mathcal{P} = \text{Frame } \mathcal{A}$,
2. or $\mathcal{A}$ is special and Frame $\mathcal{A}' = \mathcal{A}'/\mathcal{P} = (\mathcal{A}/\mathcal{P})^{*(\tilde{\mathcal{P}}/\mathcal{P})}$, where $\tilde{\mathcal{P}}$ denotes the canonical modular partition of $\tilde{\mathcal{A}}$.

Alternative (1) corresponds to several difference classes, and (2) to a single class. In both cases $\mathcal{A}'$ is robust of prime kind and has the same modular components as $\mathcal{A}$.

1. If $\mathcal{A}$ is $F_3$-reconstructible, then its modular components are $F_3$-reconstructible.
2. Conversely, assuming that each modular component is $F_3$-reconstructible, $\mathcal{A}$ is reconstructible if and only if it fails to be special or $\tilde{\mathcal{A}}$ is robust of prime kind (which for example follows from $\mathcal{A}$ being arc-connected and peak-free) and $\mathcal{A}$ is isomorphic to the pseudo-dual $\mathcal{A}^{*\tilde{\mathcal{P}}}$, obtained from it by reversing its pseudo-frame, i.e. the arcs transverse to $\tilde{\mathcal{P}}$. 
DEFINITIONS: Say that a substitution $\mathcal{B}[\mathcal{A}_v : v \in V]$ of nonempty structures along a nonempty non-singleton structure is **specific** if it satisfies one of the following three properties:

1. $\mathcal{B}$ is constant;
2. $\mathcal{B}$ is prime non-special;
3. there is a non-selfdual label $\lambda \in \Lambda$ for which one of the two properties below is satisfied, in which case the substitution is said to be $\lambda$-**specific**:
   1. $\mathcal{B}$ is $\lambda$-linear and one of the two properties below is satisfied:
      1. $\mathcal{B}$ is finite and the summands $\mathcal{A}_v$ are pairwise isomorphic $\lambda$-tournaments;
      2. between any two $\lambda$-tournament summands there lies a non-$\lambda$-tournament summand,
      2. each non-$\lambda$-tournament summand fails to have any non-coarse $\lambda$-linear quotient;
   2. $\mathcal{B}$ is prime and $\lambda$-special and the following two properties are satisfied:
      1. the summands are $\lambda$-tournaments,
      2. there exists an isomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{B}^{*\sim}$, from $\mathcal{B}$ onto its pseudodual, such that for each vertex $v \in V$ of $\mathcal{B}$, $\mathcal{A}_\varphi(v)$ is isomorphic to $\mathcal{A}_v$ (in particular $\mathcal{B}$ is self-pseudodual, i.e., isomorphic to its pseudodual).
The class of nonempty short $F_3$-reconstructible reversible $\Lambda$-2-structures is the closure of the class of singleton $\Lambda$-2-structures under specific substitutions.

Remark

For a $\lambda$-special structure $\mathcal{A}$ to be $F_3$-reconstructible it suffices that its components be $F_3$-reconstructible and that $\mathcal{A}$ be isomorphic to a particular $F_3$-binom, viz. to its pseudodual $\mathcal{A}^*\tilde{\mathcal{P}}$, which is its unique binom, besides itself, if $\mathcal{A}$ is prime. Note that the components are $\lambda$-tournaments; if the structure $\mathcal{A}$ is short then they are $F_3$-reconstructible if and only if their modules are all selfdual.
**Corollary (Reconstruction of prime structures)**

Consider a prime reversible \( \Lambda \)-2-structure \( \mathcal{A} \). If \( \mathcal{A} \) fails to be special then it has no other \( F_3 \)-hypomorphy binom than itself, in which case it is \( F_3 \)-reconstructible. If \( \mathcal{A} \) is special, then its pseudo-dual \( \mathcal{A}^* \tilde{P} \) is prime and is its unique \( F_3 \)-hypomorphy binom besides itself, in which case \( \mathcal{A} \) is \( F_3 \)-reconstructible if and only if it is self pseudo-dual, i.e., if and only if \( \mathcal{A} \cong \mathcal{A}^* \tilde{P} \). (\( \tilde{P} \) denotes the canonical pseudo-modular partition of \( \mathcal{A} \).)
Corollary (Preservation of species)

Robustness is preserved by $F_3$-hypomorphy, together with the canonical modular partition, the kind of robustness, as well as the colour of robustness in the elementary case. In particular non-robustness is preserved too, as well as primality.

In particular species is preserved by $F_3$-hypomorphy.
Reconstructible short tournament

Corollary ($F_3$-reconstructible short tournaments)

For a short tournament the following properties are mutually equivalent:

1. It is $F_3$-reconstructible.
2. Each of its inner robust modules of linear kind has a finite frame and its modular components are pairwise isomorphic, and for each inner robust module of prime kind its frame has an isomorphism onto the dual of this frame mapping each modular component to an isomorphic component.
3. Every module is selfdual.
A Λ2s is short if and only if every chain of its strong modules is finite.

**Definition**

- A Λ2s is **founded** if its collection of strong modules is well-founded for inclusion.
- A Λ2s is **co-founded** if its collection of strong modules is well-founded for the reverse of inclusion.

A Λ2s is co-founded if and only if its nonempty strong module are robust.

In particular a co-founded nonempty Λ2s is robust.
Near-short touraments of which every module is self-dual

Let $V$ denote the set of sequences $x \in \{0, 1, 2\}^\omega$, and for $K \subseteq \{0, 1, 2\}$ let $V_K$ denote the set of those sequences that are eventually constant at some element of $K$. We endow $V$ with a tournament relation $\rightarrow^i$ and each $V\{k\}$ with a tournament relation $\rightarrow^f$, defined as follows: First let $\leftarrow$ and $\rightarrow$ denote the usual oriented cycle relation and the linear ordering on $\{0, 1, 2\}$ that differ at $\{0, 2\}$:

- $0 \leftrightarrow 1 \leftrightarrow 2$ and $0 \rightarrow 1 \rightarrow 2$
For distinct $x$ and $y$ in $V$, let $\delta_i(x, y)$ denote the first integer at which they differ and then let $x \downarrow_i y := x(\delta_i(x, y))$ and $y \downarrow_i x := y(\delta_i(x, y))$ denote the first values that distinguish $x$ and $y$. Then let:

$$x \xrightarrow{i} y :\iff \begin{cases} 
\delta_i(x, y) \text{ is odd and } x \downarrow_i y \xrightarrow{i} y \downarrow_i x \\
\delta_i(x, y) \text{ is even and } x \downarrow_i y \xrightarrow{i} y \downarrow_i x 
\end{cases}$$

Claim : $(V_{\{0, 2\}}, \xrightarrow{i})$ is a non $F_3$-reconstructible co-founded tournament of which all modules are selfdual.
For distinct \( x \) and \( y \) in \( V_{\{k\}} \), let \( \delta_f(x, y) \) denote the last integer at which they differ and then let \( x \downarrow_f y := x(\delta_f(x, y)) \) and \( y \downarrow_f x := x(\delta_f(x, y)) \) denote the last values that distinguish \( x \) and \( y \). Then let:

\[
x \xrightarrow{f} y : \iff \begin{cases} 
\delta_f(x, y) \text{ is odd and } x \downarrow_f y \xrightarrow{o} y \downarrow_f x \\
\delta_f(x, y) \text{ is even and } x \downarrow_f y \xrightarrow{e} y \downarrow_f x 
\end{cases}
\]

Claim : \( (V_{\{1\}}, \rightarrow_f) \) is a non \( \mathbb{F}_3 \)-reconstructible founded tournament of which all modules are selfdual.
According to the following claims, \((V\{1\}, \rightarrow)\) and \((V\{0,2\}, \rightarrow)\) are respectively founded and co-founded tournaments failing to be \(F_3\)-reconstructible although each of their modules is selfdual:

\[
(V\{1\}, \rightarrow) \simeq_{F_3} (V\{1\}, \rightarrow) \simeq (V\{0\}, \rightarrow) \not\simeq (V\{1\}, \rightarrow)
\]

and:

\[
(V\{0,2\}, \rightarrow) \simeq_{F_3} (V\{0,2\}, \rightarrow) \simeq (V\{1,2\}, \rightarrow) \not\simeq (V\{0,2\}, \rightarrow)
\]
Claim 1:
- For the tournament structures induced by $i$, each of $V_{\{1,2\}}, V_{\{0,2\}}$ and $V_{\{0,1\}}$ is co-founded, of depth $\omega$.
- For the tournament structures induced by $f$, each of $V_{\{0\}}, V_{\{1\}}$ and $V_{\{2\}}$ is founded, of height $\omega$. Their vertex set is their only strong module that fails to be robust.

Claim 2:
- Each module of $(V_{\{0,2\}}, i)$ is isomorphic to its dual. The dual of $(V_{\{0,1\}}, i)$ is isomorphic to $(V_{\{1,2\}}, i)$.
- Each module of $(V_{\{1\}}, f)$ is isomorphic to its dual. The dual of $(V_{\{0\}}, f)$ is isomorphic to $(V_{\{2\}}, f)$.

Claim 3:
- $(V_{\{0,2\}}, i)$ is $F_3$-hypomorphic to a structure isomorphic to $(V_{\{1,2\}}, i)$.
- $(V_{\{1\}}, f)$ is $F_3$-hypomorphic to a structure isomorphic to $(V_{\{0\}}, f)$.

Claim 4:
- For the tournament structures induced by $i$, $V_{\{1,2\}}, V_{\{0,2\}}$ and $V_{\{0,1\}}$ are pairwise non-isomorphic.
- For the tournament structures induced by $f$, $V_{\{0\}}, V_{\{1\}}$ and $V_{\{2\}}$ are pairwise non-isomorphic.