Remarks on Skula spaces

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The lecture is a part of a joint work with T. Banack and W. Kubiś:

Some references


Definition 1.

For a topological space $X$, we say that a family $\mathcal{U} := \{U_x : x \in X\}$ is a clopen selector if each $U_x$ is a closed and open (clopen) subset of $X$ and if $\mathcal{U}$ satisfies:

1. $x \in U_x$ for every $x \in X$ and
2. the relation “$x < y$ if and only if $x \neq y$ and $U_x \subseteq U_y$” is irreflexive and transitive.

A Skula space $X$ is a compact 0-dimensional space having a clopen selector (so $X$ is a Priestley space).

If $\mathcal{U}$ is a clopen selector of a Skula space $X$ then:

- $\mathcal{U}$ defines the topology on the compact space $X$.
- $X$ is a (topologically) scattered space: every nonempty closed set $F$ has an isolated point (for the induced topology).
- Every closed initial subset $K$ of $X$ (in particular $U_x \cap U_y$) is a finite union of $U_z$; and thus $K$ is clopen.

So in some sense $\mathcal{U}$ is a “semi-meet-semilattice”.
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So in some sense $\mathcal{U}$ is a “semi-meet-semilattice”.
Example. $\alpha + 1$ is an ordinal space and $U_\beta := [0, \beta]$ is clopen for any $\beta$.

Basic picture of a Skula space.
- A cyan $\bullet$ point is minimal in $X$, i.e. $U_\bullet := \{x \in X : x \leq \bullet\} = \{\bullet\}$ and are isolated in $X$.
- The red $\bullet$ (or $\bullet, \bullet, \ldots$) point is of level 1 and is the unique non-isolated point of $\{x \in X : x \leq \bullet\}$. So red $\bullet$ (or $\bullet, \bullet, \ldots$) are elements of $D(X)$.
- The black $\bullet$ point at the top is of level 2 and is the unique non-isolated point of $D(X)$.

We can construct such a space of all (finite or infinite) level.
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**Example.** Let $P$ be a poset. Denote by $FS(P) \subseteq \{0, 1\}^P$ (resp. $IS(P) \subseteq \{0, 1\}^P$) the set of all final (resp. initial) subsets of $P$. Endow $FS(P)$ and $IS(P)$ of the pointwise topology. Identify $FS(P)$ and $IS(P)$ by $A \mapsto P \setminus A$.

**Fact 2.**

Let $P$ be a partial ordering. The following are equivalent.

(i) $P$ is a well-quasi-ordering (well-founded with no infinite antichain).

(ii) $\langle IS(P), \subseteq \rangle$ (i.e. $\langle FS(P), \supseteq \rangle$) is well-founded.

(iii) Any nonempty final subset $K$ of $P$ is finitely generated by a finite subset $\sigma_K$ of $P$.

**Proposition 3.**

Let $P$ be a well-quasi ordering and let $IS(P)$ endowed with the pointwise topology.

Then $IS(P)$ is a Skula space, and thus $IS(P)$ is a scattered space.
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Invariants for a Skula space.

Denote by $D(Y)$ the set of non-isolated points of $Y$. Moreover set $D^0(X) = X$ and $D^\alpha(X) = D\left(\bigcap_{\beta < \alpha} D^\beta(X)\right)$.

Let $\mathcal{U}$ be a clopen selector of a Skula space $X$. Then

1. $\mathcal{U}$ is well-founded. Therefore $\langle X, \subseteq \rangle$ has a (well-founded) rank:
   
   $rk_{WF}X(x) = \sup\{rk_{WF}X(y) : y < x\}$.
   
   So $rk_{WF}X(x) = 0$ if and only if $x$ is minimal, i.e. $U_x = \{x\}$. Moreover $rk_{WF}(X) := \sup_{x \in X} rk_{WF}X(x)$.
   
   By compactness $rk_{WF}(X) = \sup_x rk_{WF}X(x)$ is the last (ordered) derivative is nonempty and finite.

2. $X$ is compact and scattered. Therefore $X$ has a (Cantor-Bendixson) height:
   
   $ht_{CB}X(x) = \gamma$ iff $x \in D^{\gamma+1}(X) \setminus D^{\gamma}(X)$.
   
   By compactness $ht_{CB}(X) = \sup_x ht_{CB}X(x)$ is the last (topological) derivative $Endpt(X)$ is nonempty and finite.

We have $ht_{CB}X(x) \leq rk_{WF}X(x)$ for $x \in X$. 
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1. $\mathcal{U}$ is well-founded. Therefore $\langle X, \subseteq \rangle$ has a (well-founded) rank:

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**Hyperspace $H(X)$ of a Skula space $X$.**

We define the Vietoris hyperspace $H(X)$ over a Skula space $X$ as follows:

- $H(X)$ is the set of all nonempty closed initial subsets of $\langle X, \leq \rangle$. Therefore $\mathcal{U} \subseteq H(X)$.

- For $F, G \in H(X)$, we set $F \leq G$ if and only if $F \subseteq G$.

- The topology on $H(X)$ is the topology generated by the sets
  
  $$U^+ := \{ K \in H(X) : K \subseteq U \}$$

  declared to be clopen where $U$ is any clopen initial subset in $X$.

So $V^- := \{ K \in H(X) : K \cap V \neq \emptyset \}$ is clopen in $H(X)$ if $V$ is clopen final in $X$.

**Theorem 4.**

Let $X$ be a Skula space. Then $H(X)$ is a Skula space.

Main order property ($A, B \in H(X)$):

- $H(X)$ is a continuous join-semilattice where $A \vee B := A \cup B$. 
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Main order property ($A, B ∈ H(X)$):

- $H(X)$ is a continuous join-semilattice where $A ∨ B := A ∪ B$. 
A space \( X \) is canonically Skula if \( X \) has a clopen selector 
\[ \mathcal{U} := \{ U_x : x \in X \} \] 
such that for \( x \in X \) 
\[ D^{\alpha_x}(U_x) = \{ x \} \] 
for some \( \alpha_x \) and \( \mathcal{U} \) is called a canonical clopen selector.

**Theorem 5.**

Let \( X \) be a canonical Skula space. Then

1. \( r_{WF}(x) = h_{CB}(x) (= \alpha_x) \) for \( x \in X \).
2. \( H(X) \) is canonically Skula.

Let \( P \) be a well-quasi-ordering (wqo). We have seen that \( IS(P) \), with the pointwise topology, is a Skula space.

**Main Question 6.**

1. Let \( P \) be a well-quasi-ordering (w.q.o.). Is \( IS(P) \) canonically Skula?
2. Let \( P \) be a better-quasi-ordering (b.q.o.). Is \( IS(P) \) canonically Skula?
(For b.q.o. see Nash-Williams.)
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A space $X$ is canonically Skula if $X$ has a clopen selector
$\mathcal{U} := \{ U_x : x \in X \}$ such that for $x \in X$ $D^{\alpha_x}(U_x) = \{ x \}$ for some $\alpha_x$ and $\mathcal{U}$ is called a canonical clopen selector.

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Let $X$ be a canonical Skula space. Then

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A computation of height and rank on canonical Skula space.

Proposition 7. Let $\mathcal{U} := \{U_x : x \in X\}$ be a canonical selector for $X$ and let $x \in X$ (so $\mathcal{U} \subseteq H(X)$). Then

$$\text{rk}_{WF}(U_x) = \text{rk}_{WF}(x) = \text{ht}_{CB}(x) = \text{ht}_{CB}(U_x)$$

and

1. If $\text{rk}_{WF}(x) = 0$ then $\text{ht}_{CB}(H(X))(U_x) = 0$.
2. If $\text{rk}_{WF}(x) = 1$ then $\text{ht}_{CB}(H(X))(U_x) = 1$.
3. If $\text{rk}_{WF}(x) = 1+\alpha \geq 2$ then $\text{ht}_{CB}(H(X))(U_x) = \omega^\alpha$.

Application. Let $U_\sigma := \bigcup_{x \in \sigma} U_x$ where $\sigma = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ is an antichain of $X$ satisfying

$$
\begin{align*}
\text{ht}_{CB}(x_0) &= 0 \\
\text{ht}_{CB}(x_1) &= 1 = \text{ht}_{CB}(x_2) \\
\text{ht}_{CB}(x_3) &= 2 \\
\text{ht}_{CB}(x_4) &= 10 = \text{ht}_{CB}(x_5) \\
\text{ht}_{CB}(x_6) &= \omega + 7 \\
\text{ht}_{CB}(x_7) &= 3.
\end{align*}
$$

By Proposition 7 and a theorem of Telgàsky, we have:

$$\text{ht}_{CB}(H(X))(U_\sigma) = \omega^{\omega+7} + \omega^9 \cdot 2 + \omega^2 + \omega + 2.$$
Thanks