Monotonic Computation Rules for Nonassociative Calculus

Miguel Couceiro  
*Université de Lorraine, CNRS*  
*Inria N.G.E., LORIA*

Michel Grabisch  
*Paris School of Economics, Université Paris 1*  
*Centre d’Economie de la Sorbonne*
L countable, totally ordered, with bottom element 0

\(-L := \{-a : a \in L\}\) symmetric copy with reversed order;
\(\tilde{L} := L \cup (-L) \setminus \{-0\}\) its bipolar extension
(typically, \(L = \mathbb{N}\), \(\tilde{L} = \mathbb{Z}\))

**Aim:** define an extension on \(\tilde{L}\) of the maximum \(\lor\) on \(L\), denoted \(\ominus\), to mimic addition on \(\mathbb{R}\):

\((C1)\) \(\ominus\) coincides with the maximum on \(L^2\);
\((C2)\) \(a \ominus (-a) = 0\) for every \(a \in \tilde{L}\);
\((C3)\) \(- (a \ominus b) = (-a) \ominus (-b)\) for every \(a, b \in \tilde{L}\).

Such a task is doomed to fail, as already (C1) and (C2) imply the failure of associativity. With \(L = \mathbb{N}\):

\((2 \ominus 3) \ominus (-3) = 3 \ominus (-3) = 0;\quad 2 \ominus (3 \ominus (-3)) = 2 \ominus 0 = 2\)
G. (2004) proved that the operation $\vee$ satisfying (C1), (C2), (C3) and being associative on the largest domain is defined by:

$$
a \vee b = \begin{cases} 
- (|a| \lor |b|) & \text{if } b \neq -a \text{ and } |a| \lor |b| = -a \text{ or } = -b \\
0 & \text{if } b = -a \\
|a| \lor |b| & \text{otherwise.}
\end{cases}
$$

It is called the **symmetric maximum**.

More precisely, $\vee$ is associative on an expression involving at least 3 nonzero elements $a_1, \ldots, a_n \in \tilde{L}$ if and only if

$$
\bigvee_{i=1}^{n} a_i \neq - \bigwedge_{i=1}^{n} a_i
$$

Sequences fulfilling this condition were referred to as **associative** in (C. and G., 2013).
Introduction (3/4)

Solution for the associativity problem: define *rules of computation*, i.e., systematic ways of putting parentheses (G., 2004). For example:

1. aggregate separately positive and negative terms, then compute their symmetric maximum

\[\oplus(3, 2, -3, 1, -3, -2, 1) = (3 \oplus 2 \oplus 1 \oplus 1) \oplus ((-3) \oplus (-3) \oplus (-2)) \]
\[= 3 \oplus (-3) = 0.\]

2. aggregate first extremal opposite terms to cancel them, till there is no more extremal opposite terms

\[\oplus(3, 2, -3, 1, -3, -2, 1) = (3 \oplus (-3)) \oplus (2 \oplus 1 \oplus (-3) \oplus (-2) \oplus 1) \]
\[= 0 \oplus (-3) = -3.\]

3. the same as above, but first aggregate these extremal opposite terms

\[\oplus(3, 2, -3, 1, -3, -2, 1) = (3 \oplus ((-3) \oplus (-3))) \oplus (2 \oplus (-2)) \oplus (1 \oplus 1) \]
\[= (3 \oplus (-3)) \oplus 0 \oplus 1 = 0 \oplus 1 = 1.\]
Solution for the associativity problem: define *rules of computation*, i.e., systematic ways of putting parentheses (G., 2004). For example:

1. aggregate separately positive and negative terms, then compute their symmetric maximum

\[ \ominus(3, 2, -3, 1, -3, -2, 1) = (3 \ominus 2 \ominus 1 \ominus 1) \ominus((-3) \ominus(-3) \ominus(-2)) \]
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3. the same as above, but first aggregate these extremal opposite terms

\[ \ominus(3, 2, -3, 1, -3, -2, 1) = (3 \ominus((-3) \ominus(-3))) \ominus(2 \ominus(-2)) \ominus(1 \ominus 1) \]
\[ = (3 \ominus(-3)) \ominus 0 \ominus 1 = 0 \ominus 1 = 1. \]

This amounts to deleting terms in a certain way in the sequence
For each computation rule $R$, we obtain a corresponding \textit{aggregation operator} $\bigodot_R$, defined for any sequence $\sigma$ of $\tilde{L}^*$ by

$$\bigodot_R(\sigma) = \bigodot(R(\sigma))$$

Taking $\tilde{L} = \mathbb{Z}$, $\bigodot_R$ can be seen as a particular \textit{integer mean} or \textit{$\mathbb{Z}$-mean} (Bennett, Holland and Székely, 2014), (C. and G., 2017).

\textit{Monotonicity} (nondecreasingness) is a fundamental property for aggregation operators in applications.

\textbf{Main result of the paper: characterization of monotonic computation rules}, i.e., which lead to monotonic aggregation operators.
We consider that $L$ is countably infinite, i.e., $\tilde{L}$ is isomorphic to $\mathbb{Z}$. Then $\tilde{L}^*$ is the set of all finite sequences of integers, including the empty sequence $\varepsilon$.

As $\bigodot$ is commutative, we order sequences in decreasing magnitude and adopt the following notation:

$$
\sigma = (n_1, \ldots, n_1, -n_1, \ldots, -n_1, \ldots, n_q, \ldots, n_q, -n_q, \ldots, -n_q)
$$

with $n_1 \geq \cdots \geq n_q$

Let $\mathcal{S}$ denote the set of all integer sequences in this formalism, including the empty sequence, and let $\mathcal{S}_0$ be the subset of all nonassociative sequences.

Note: $\sigma \in \mathcal{S}_0$ iff $p_1 > 0$ and $m_1 > 0$. 
Computation rules

We define 5 elementary rules acting on any sequence $\sigma$:

1. Elementary rule $\rho_1$: if $p_1 > 1$ and $m_1 > 0$, then $p_1$ is changed to $p_1 = 1$;
   
   Ex: $\rho_1(3, 3, 3, -3, -3, 2, -2, 1) = (3, -3, -3, 2, -2, 1)$

2. Elementary rule $\rho_2$: same as in (1) with $p_1, m_1$ exchanged;
   
   Ex: $\rho_1(3, 3, 3, -3, -3, 2, -2, 1) = (3, 3, -3, 2, -2, 1)$

3. Elementary rule $\rho_3$: if $p_1 > 0$, $m_1 > 0$, the pair $(p_1, m_1)$ is changed into $(p_1 - c, m_1 - c)$, where $c = p_1 \wedge m_1$;
   
   Ex: $\rho_1(3, 3, 3, -3, -3, 2, -2, 1) = (3, 2, -2, 1)$

4. Elementary rule $\rho_4$: if $p_1 > 0$, $m_1 > 0$, and if $p_2 > 0$, then $p_2$ is changed into $p_2 = 0$;
   
   Ex: $\rho_1(3, 3, 3, -3, -3, 2, -2, 1) = (3, 3, 3, -3, -3, -2, 1)$

5. Elementary rule $\rho_5$: same as in (4) with $m_2$ replacing $p_2$.
   
   Ex: $\rho_1(3, 3, 3, -3, -3, 2, -2, 1) = (3, 3, 3, -3, -3, 2, 1)$

Note: Elementary rules delete terms only in nonassociative sequences, and leave the associative ones invariant.
A **(well-formed) computation rule** $R$ is a word built with the alphabet $\{\rho_1, \ldots, \rho_5\}$, i.e., $R \in L(\rho_1, \ldots, \rho_5)$, such that $R(\sigma) \in \mathcal{G} \setminus \mathcal{G}_0$ for all $\sigma \in \mathcal{G}$.

The set of (well-formed) computation rules is denoted by $\mathcal{R}$.

**Examples:**

1. $\langle \cdot \rangle^+ = (\rho_4 \rho_5)^* \rho_1 \rho_2 \rho_3$
   
   $\langle 3, -3, -3, 2, -2, 1 \rangle^+ = \varepsilon$

2. $\langle \cdot \rangle_0 = \rho_3^*$.
   
   $\langle 3, -3, -3, 2, -2, 1 \rangle_0 = (-3, 2, -2, 1)$

3. $\langle \cdot \rangle_- = (\rho_1 \rho_2 \rho_3)^*$
   
   $\langle 3, -3, -3, 2, -2, 1 \rangle_- = (1)$

We define $\bigodot_R := \bigodot \circ R$, with the convention $\bigodot_R(\varepsilon) = 0$ and $\bigodot_R(a) = a$ for all $a \in \tilde{L}$. 

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Let $R, R' \in \mathcal{R}$ and, for each sequence $\sigma = (a_i)_{i \in I}$, let $J_\sigma \subseteq I$ and $J'_\sigma \subseteq I$, be the sets of indices of the terms in $\sigma$ deleted by $R$ and $R'$, respectively.

We write $R \leq R'$ if for all sequences $\sigma \in \mathcal{S}$ we have $J_\sigma \supseteq J'_\sigma$.

Then $(\mathcal{R}, \leq)$ is an uncountable poset, with a unique bottom element $\langle \cdot \rangle^+$, and infinitely many maximal elements (e.g., $\langle \cdot \rangle_0$).

We conjecture that it is a inf-semilattice.
Factorization of computation rules

Each computation rule $R \in \mathcal{R}$ can be put in \textit{factorized irredundant form (FIF)}:

1. \textbf{Factorization}: $R$ can be factorized into a composition
   
   $$R = T_1 T_2 \cdots T_i \cdots$$

   where each term has the form $T_i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$, with $\omega_i \in \mathcal{L}(\{\rho_4, \rho_5\})$ (possibly empty), and $a_i, b_i \in \{0, 1\}$.

2. \textbf{Simplification}: Suppose that in (1) there exists $j \in \mathbb{N}$ such that $\omega_j = \omega \rho_4^{\ast}$ or $\omega \rho_5^{\ast}$ for some $\omega \in \mathcal{L}(\{\rho_4, \rho_5\})$, or that $\rho_4$ and $\rho_5$ alternate infinitely many times in $\omega_j$. Let

   $$k_1 = \min\{j : \omega_j = \omega \rho_4^{\ast} \text{ or } \omega \rho_5^{\ast}\}, \quad \text{and}$$

   $$k_2 = \min\{j : \rho_4 \text{ and } \rho_5 \text{ alternate infinitely many times in } \omega_j\}.$$

   - If $k_1 < k_2$, then $R \sim T_1 \cdots T_{k_1}$.
   - Otherwise, $k_2 \leq k_1$, and $R \sim T_1 \cdots T'_{k_2}$, where

     $$T'_{k_2} = (\rho_4 \rho_5)^{\ast} \rho_1^{a_{k_2}} \rho_2^{b_{k_2}} \rho_3.$$
Monotonic computation rules

Not all computation rules are monotonic!

Examples:

1. $\langle \cdot \rangle_0$ and $\langle \cdot \rangle^\dagger$ are monotonic

2. $\langle \cdot \rangle = (\rho_1 \rho_2 \rho_3)^*$ is not:

$$\bigodot \langle \cdot \rangle (5, -5, -5, 4, 3) = 4 \quad \text{whereas} \quad \bigodot \langle \cdot \rangle (5, -5, -4, 4, 3) = 3.$$ 

Note: $\bigodot$ is monotonic on $\tilde{L}^2$, hence all $\bigodot_R$ are monotonic on associative sequences.
Computation rules with a single term

**Lemma**

If $R$ has the form $(\rho_4 \rho_5)^* \rho_1^a \rho_2^b \rho_3$ then $\bigotimes_R$ is monotonic.

**Lemma**

Let $R \in \mathcal{R}$ be in FIF.

1. Suppose that $R$ has the form $\omega \rho_1^a \rho_2^b \rho_3$ for $\omega = \omega' \rho^*_4$ with $\omega' \in \mathcal{L}(\rho_4, \rho_5)$. Then $R$ is monotonic if and only if $(a, b) = (a, 0)$, for $a \in \{0, 1\}$, and $\omega' = \varepsilon$.

2. Suppose that $R$ has the form $\omega \rho_1^a \rho_2^b \rho_3$ for $\omega = \omega' \rho^*_5$ with $\omega' \in \mathcal{L}(\rho_4, \rho_5)$. Then $R$ is monotonic if and only if $(a, b) = (0, b)$, for $b \in \{0, 1\}$, and $\omega' = \varepsilon$. 
2 auxiliary results:

**Lemma**

Suppose that $\ominus_R$ is not monotonic, and let $T \in \mathcal{L}(\rho_1, \ldots, \rho_5)$ such that $TR \in \mathcal{R}$ be in FIF. Then $\ominus_{TR}$ also is not monotonic.

**Lemma**

Let $R \in \mathcal{R}$ be in FIF, and let $T = \rho_3^k R$ with $k \geq 1$. Then, $\ominus_R$ is monotonic if and only if $\ominus_T$ is monotonic.
Lemma

Let $R = T^1 T^2 \cdots$ be in FIF where $T^i = \omega_i \rho_1^{a_i} \rho_2^{b_i} \rho_3$. If there exists $k$ such that

- $(a_k, b_k) = (1, 0)$ and $\omega_k \neq \rho_4^*, (\rho_4 \rho_5)^*$,
- $(a_k, b_k) = (0, 1)$ and $\omega_k \neq \rho_5^*, (\rho_4 \rho_5)^*$, or
- $(a_k, b_k) = (1, 1)$ and $\omega_k \neq \rho_4^*, \rho_5^*, (\rho_4 \rho_5)^*$,

then $\bigvee_R$ is not monotonic.

Lemma

Suppose $R = T^1 T^2 \cdots$ is in FIF, and that no term contains $\rho_1$ nor $\rho_2$. If there is $k \geq 1$ such that $\omega_k$ in $T^k$ is of the AFT type or equal to $\rho_4^\alpha$ or $\rho_5^\beta$, then $\bigvee_R$ is not monotonic.
Summarizing the above results, we get a characterization of monotonic computation rules:

**Theorem**

Let $R \in \mathbb{R}$ be in FIF. Then $\ominus_R$ is monotonic if and only if either

1. $R = \rho_3^*$, or

2. $R = \rho_3^k T$, where $T = \omega \rho_1^a \rho_2^b \rho_3$ satisfies the following conditions
   - if $(a, b) = (1, 0)$, then $\omega = \rho_4^*$ or $(\rho_4 \rho_5)^*$,
   - if $(a, b) = (0, 1)$, then $\omega = \rho_5^*$ or $(\rho_4 \rho_5)^*$,
   - if $(a, b) = (1, 1)$, then $\omega = (\rho_4 \rho_5)^*$,
   - if $(a, b) = (0, 0)$, then $\omega = \rho_4^*, \rho_5^*, (\rho_4, \rho_5)^*$. 

Monotonic Computation Rules