Unstable graphs and packing into fifth power

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summary

1. Definitions and results
2. Characterisation of the unstable graphs
3. Packing of an unstable graph into it’s fifth power
All graphs considered in this paper are finite, undirected, without loops or multiple edges.
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For a graph $G$, we will use $V(G)$, and $E(G)$, to denote its vertex and edge sets respectively.

The $k$–th power $G^k$ of a graph $G$ is the graph obtained from $G$ by adding an edge between all pairs of vertices of $G$ with distance at most $k$. 
Packing of graphs

Definition

Let $G_1, \ldots, G_k$ be a $k$ graphs of order $n$. We say that there is a packing of $G_1, \ldots, G_k$ (into the complete graph $K_n$) if there exist injections $\alpha_i : V(G_i) \rightarrow V(K_n)$, $i = 1, \ldots, k$, such that $\alpha_i^*(E(G_i)) \cap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^* : E(G_i) \rightarrow E(K_n)$ is the one induced by $\alpha_i$. 

Example

Figure – A packing of two copies of $P_4 \cup P_3$ into $K_7$. 
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\[ \text{Figure} \] – A packing of two copies of $P_4 \cup P_3$ into $K_7$. 
Unstable graphs

Indecomposable graphs

In a graph $G$, a subset $M$ of the vertex set $V$ is a module (or interval, clan) of $G$ if every vertex outside $M$ is adjacent to all or none of $M$. The empty set, the singleton sets, and the full set of vertices are trivial modules. A graph is indecomposable (or prime) if all its modules are trivial. In the opposite case, we will say that $G$ is decomposable.
Unstable graphs

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Unstable graphs

Let $G$ be an indecomposable graph. We say that $e \in E$ is removable edge if $G - e$ is indecomposable. The graph $G$ is said to be unstable if it has no removable edges. Hence $G$ is unstable if the removal of any edge $e \in E$ creates a nontrivial module in $G - e$. 
Main result

Theorem

Let $G$ be an unstable graph. Then there exists a 2-placement $\sigma$ of $G$ such that $\sigma(G) \subseteq G^5$. 
Definitions and notations

\[ \text{Leaf}(G) = \{ x \in V, \ x \text{ is a leaf} \}. \]
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If \(x\) is a leaf, then the unique edge \(e\) incident with \(x\) is called pendant edge.
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An edge \(e\) is a bridge if \(G - e\) is disconnected graph, otherwise, \(e\) is a non-bridge. A bridge is said to be proper if it is not a pendant edge.
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Nob(G) = \{e \in E, e \text{ is not a bridge}\}

Isl(G) = \{x \in V, x \text{ is an island}\}
Let $G$ be an indecomposable graph. We say that $x \in V(G)$ is an \textit{inside vertex}, if there exists a non-bridge $e = xy \in Nob(G)$ such that $x \in X$ for a nontrivial module $X$ of $G - e$. On the other hand, if there exists a non-bridge $e = xy \in Nob(G)$ such that $x \notin X$ for a nontrivial module $X$ of $G$, then $x$ is called an \textit{outside vertex}. 
Let $G$ be an indecomposable graph. We say that $x \in V(G)$ is an *inside vertex*, if there exists a non-bridge $e = xy \in Nob(G)$ such that $x \in X$ for a nontrivial module $X$ of $G - e$. On the other hand, if there exists a non-bridge $e = xy \in Nob(G)$ such that $x \notin X$ for a nontrivial module $X$ of $G$, then $x$ is called an *outside vertex*.

$$Out(G) = \{ x \in V ; x \text{ is an outside vertex} \}.$$  
$$Ins(G) = \{ x \in V ; x \text{ is an inside vertex} \}.$$
Definitions and notations

Pendant component

Let us call a subgraph $H$ a *pendant component* of a graph $G$ if $H$ is a connected component of a graph $G'$, which is obtained by removing from $G$ all its proper bridges. If $G$ is its pendant component, then it is called a pendant graph. In this case, $G$ has no proper bridges.
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$B_{IO} -$ graph

We call a $B_{IO} -$ graph a bipartite indecomposable graph $G$ of order $n \geq 5$, with a bipartition $\{I, O\}$ such that for all $y \in I$ and for all $k -$ subset $X_k$ of $N_G(y)$, $1 \leq k \leq d_G(y) - 1$, there exists a vertex $v_k \in I \setminus \{y\}$ such that $N_G(v_k) = X_k$
Definitions and notations

Example

Figure – A $B_{10}$ – graph with 10 vertices.
Definitions and notations

**Example**

![Figure - A $B_{IO}$ graph with 10 vertices.](image)

**Lemma 1**

Let $G$ be a $B_{IO}$ graph. Then $G$ is unstable.
Lemmas

Lemma 2

Let $G$ be a $B_{IO} - graph$ and $e = xy \in E(G)$ be a no proper bridge, such that $x \in O$ and $y \in I$. Then, $x \in Out(G)$ and either $y \in Leaf(G)$ or $y \in Ins(G)$. 
Lemmas

Lemma.2
Let $G$ be a $B_{IO} - graph$ and $e = xy \in E(G)$ be a no proper bridge, such that $x \in O$ and $y \in I$. Then, $x \in \text{Out}(G)$ and either $y \in \text{Leaf}(G)$ or $y \in \text{Ins}(G)$.

Lemma.3
Let $G$ be a pendant graph. If $G$ is unstable, then $G$ is a $B_{IO} - graph$. 
Lemmas

Lemma.2
Let $G$ be a $B_{IO}$ graph and $e = xy \in E(G)$ be a no proper bridge, such that $x \in O$ and $y \in I$. Then, $x \in Out(G)$ and either $y \in Leaf(G)$ or $y \in Ins(G)$.

Lemma.3
Let $G$ be a pendant graph. If $G$ is unstable, then $G$ is a $B_{IO}$ graph.

Theorem
Let $G$ be an indecomposable graph. Then $G$ is unstable if and only if each pendant component of $G$ with at least two vertices is either a $B_{IO}$ graph or an edge.
Definitions and notations

The partition $\mathcal{P}$

Consider an unstable graph $G$. We denote by $C_1, C_2, \ldots, C_r$, $r \geq 1$, the pendant components of $G$ such that $|C_i| \geq 5$. Given an integer $j \in \{1, \ldots, r\}$, we define a partition $\mathcal{P} = \{X_1, X_2, \ldots, X_p\}$ on the vertex set of $\text{Out}(C_j)$, such that:

1. for all $i \in \{1, \ldots, p\}$, $|X_i| \in \{2, 3\}$,
2. if $|X_i| = 2$ there exists a vertex $y \in \text{Ins}(C_j)$ such that $X_i \subseteq N_{C_j}(y)$,
3. if $|X_i| = 3$ there exist two vertices $y, z \in \text{Ins}(G)$ such that $X_i \subseteq N_{C_j}(y) \cup N_{C_j}(z)$ (See Figure. 2).
Definitions and notations

Example

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example_partition.png}
\caption{An example of a partition $\mathcal{P}(O)$.}
\end{figure}
Definitions and notations

Let $x \in C_j$, $1 \leq j \leq r$. We say that $x$ is a \textit{representative vertex} of $G$ if $x$ is incident with a proper bridge. We denote by

$$R(G) = \{ x \in V(G), x \text{ is a representative vertex} \}$$

Note that $R(G) \subseteq \text{Out}(G)$. 
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Note that $R(G) \subseteq Out(G)$.

Let $V' = R(G) \cup Isl(G)$. The graph $G' = G[V']$ is called the representative graph of $G$. Note that

1. Each connected component of $G'$ is a tree.
2. Let $T_1, T_2, \ldots, T_q$, $q \geq 1$, be the connected components of $G'$. Then for all $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, q\}$, $|C_i \cap T_j| \leq 1$. 
Lemmas

Lemma 1
Let $G$ be a pendant graph of order $n \geq 5$. If $G$ is unstable, then there exists a $2$ – placement $\sigma_C$ on $V(G)$ such that $\sigma_C(G) \subseteq G^5$. 
Lemmas

Lemma 1
Let $G$ be a pendant graph of order $n \geq 5$. If $G$ is unstable, then there exists a 2−placement $\sigma_C$ on $V(G)$ such that $\sigma_C(G) \subseteq G^5$.

Lemma 2
Given an unstable graph $G$ of order $n \geq 5$, consider $G'$ its representative graph. If $G'$ is an edge, then there exists a 2−placement $\sigma_{S_2}$ on $V(G)$ such that $\sigma_{S_2}(G) \subseteq G^5$. 
Lemma.3

Given an unstable graph $G$ of order $n \geq 5$, consider $G'$ it's representative graph. If $G'$ is a star $S_p$, $p \geq 3$, then there exists a $2$-placement $\sigma_{S_2}$ on $V(G)$ such that $\sigma_{S_2}(G) \subseteq G^5$. 
Lemmas

Lemma 3
Given an unstable graph $G$ of order $n \geq 5$, consider $G'$ its representative graph. If $G'$ is a star $S_p$, $p \geq 3$, then there exists a $2$-placement $\sigma_{S_2}$ on $V(G)$ such that $\sigma_{S_2}(G) \subseteq G^5$.

Lemma 4
Given an unstable graph $G$. Consider $G'$ its representative graph. If $G'$ is a tree $U$ such that $\text{Diam}(U) \geq 3$, then there exists a $2$-placement $\sigma_U$ on $V(G)$ such that $\sigma_U(G) \subseteq G^5$. 
Idea of proof

Claim 1

If each pendant component of $G$ is either a singleton or an edge, then there exists a 2-placement $\sigma$ such that $\sigma(G) \subseteq G^3$. 
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Claim 1
If each pendant component of $G$ is either a singleton or an edge, then there exists a 2-placement $\sigma$ such that $\sigma(G) \subseteq G^3$.

Claim 2
If $G$ is a pendant graph, then there exists a 2-placement $\sigma$ such that $\sigma(G) \subseteq G^5$. 
Idea of proof

Claim 1
If each pendant component of $G$ is either a singleton or an edge, then there exists a $2$–placement $σ$ such that $σ(G) ⊆ G^3$.

Claim 2
If $G$ is a pendant graph, then there exists a $2$–placement $σ$ such that $σ(G) ⊆ G^5$.

Now, we shall assume that we can apply neither Claim 1 nor Claim 2 to the graph $G$. Under this assumption, we will define a $2$–placement $σ$ on $V(G)$ such that $σ(G) ⊆ G^5$. 
Thank You !